

21st Century Skills Map

DESIGNED IN COOPERATION WITH THE NATION'S MATH EDUCATORS

The Partnership for 21st Century Skills (P21) has forged alliances with key national organizations representing the core academic subjects, including Social Studies, English, Math, Science, Geography, World Languages and the Arts. These collaborations have resulted in the development of 21st Century Skills Maps that illustrate the essential intersection between core subjects and 21st Century Skills.

The Partnership advocates for the integration of 21st Century Skills into K-12 education so that students can advance their learning in core academic subjects.

Developed through a year-long collaborative process, this map reflects the collective effort of Mathematics professors, teachers and thought leaders, and illustrates the integration of Mathematics and 21st Century Skills. It will provide educators, administrators and policymakers with concrete examples of how 21st Century Skills can be integrated into core subjects, and how other subject areas can link successfully with mathematics.

A 21st Century Skills

B Skill Definition

An example from the Math Skills Map illustrates sample outcomes for teaching Communication and Collaboration.

The screenshot shows a page titled "Communication & Collaboration" under the heading "INFORMATION, MEDIA & TECHNOLOGY LITERACY". It features three columns for "4th Grade", "8th Grade", and "12th Grade".

- Common Core State Standards Alignment:** As noted in the P21 Common Core Tools, Communication and Collaboration skills are most explicitly aligned with the following Common Core State Standards' mathematical practices:
 - Construct viable arguments and critique the reasoning of others
 - Attend to precision
 - Look for and express regularity in repeated reasoning
- 4th Grade:**
 - OUTCOME:** Students articulate mathematical thoughts and ideas using oral and written communication skills. Using abstract and quantitative reasoning with attention to precision, they construct viable arguments and analyze others' reasoning.
 - EXAMPLE:** Students pair up and play "The Factor Game," which challenges players to find factors of numbers on a gameboard. In an online version, available at <http://illuminations.nctm.org/activitydetail.aspx?id=11>, the gameboard can include numbers up to 30, 49, or 100. Students can start with the simplest version, try to identify the mathematics that leads to a winning strategy, and then test their strategies using larger numbers. Students then present and compare their playing strategies with their classmates' approaches to determine optimal strategies. As a "Family Math Night," pairs of students present the game to their family members and explain strategies for winning.
- 8th Grade:**
 - EXAMPLE:** Working in small groups, students compare the cost of buying a hybrid versus a non-hybrid car. Students research pricing and fuel miles-per-gallon estimates for comparable models of hybrids and non-hybrids. They factor in average local gas prices over x years. Using their knowledge of linear functions, students then analyze the overall cost of the hybrid versus the non-hybrid vehicle over x number of years, using assumptions regarding the average price of gas and how many miles the car will be driven each year. Students analyze how changing the number of years or number of miles to be driven would affect the outcome. Each group presents its results and conclusions to the class.
- 12th Grade:**
 - EXAMPLE:** As a joint activity with a social studies class, students study the history of how seats have been allocated in the U.S. House of Representatives. They analyze the mathematics behind different plans and underlying socio-political issues, such as the effects on small versus large states or rural versus urban populations. Students then solve various "fair division" problems for their city council or state house of representatives using different plans. Finally, students hold a debate in which students advocate for different plans, considering both the mathematical and social issues that go into the allocation of seats.

Interdisciplinary themes are indicated by icons: Civic Literacy (top right) and Financial Literacy (middle right).

C Interdisciplinary Theme

D Sample Student Outcome/Examples



PARTNERSHIP FOR 21ST CENTURY SKILLS

Introduction

Employers and educators agree that changes in the global economy require that students entering college and the workforce leave the K-12 education system with an advanced level of proficiency in mathematics and a mastery of key mathematics concepts.

P21 believes that one of the most important ways to enable students to achieve this mastery is to fuse mathematical content and mathematical practices with the 21st Century Skills outlined in P21's Framework for 21st Century Learning.

Fusing a core subject like mathematics with 21st Century Skills makes teaching and learning more engaging, more relevant and more rigorous, ensuring that a greater number of students have an advanced level of understanding and ability in mathematics.

The working group that developed learning outcomes and instructional examples for this project identified learning expectations for grades 4, 8 and 12. These outcomes and examples are organized and grouped according to the P21 Framework for 21st Century Learning and explained in greater detail in this section.

Learning and Innovation Skills

Learning and innovation skills are increasingly recognized as those that distinguish students who are prepared for more complex life and work environments in the 21st century from those who are not. A focus on the 4C's of Creativity, Critical thinking, Communication and Collaboration is essential to prepare students for the future.

Creativity and Innovation: Students use a wide range of techniques to create new and worthwhile ideas, elaborate, refine, analyze and evaluate their own ideas in order to improve and maximize creative efforts, and demonstrate originality and inventiveness, in both an individual as well as group settings.

Critical Thinking and Problem Solving: Students reason effectively, use systems thinking and understand how parts of a whole interact with each other. They make judgments, decisions and solve problems in both conventional and innovative ways.

Communication and Collaboration: Students know how to articulate thoughts and ideas effectively using oral, written and nonverbal communication. They listen effectively to decipher meaning, such as knowledge, values, attitudes and intentions, and use communication for a wide range of purposes in diverse teams and environments.

P21 COMMON CORE TOOLKIT

P21 has created the Common Core Toolkit to align the P21 Framework with the Common Core State Standards (CCSS), a state led initiative to establish college and career standards in Mathematics and English Language Arts. The CCSS support the integration of 21st Century Skills as part of mathematics pedagogy and can offer creative ways to deepen student content knowledge and support individualized learning. Where appropriate, this document highlights connections between these examples and the CCSS. For more information on the 21st Century Skills and the Common Core, please visit <http://www.p21.org/tools-and-resources/publications/p21-common-core-toolkit>.



Introduction (continued)

Information, Media and Technology Skills

In the 21st century, we live in a technology and media-suffused environment, marked by: 1) access to an abundance of information, 2) rapid changes in technology tools, and 3) the ability to collaborate and make individual contributions on an unprecedented scale. To be effective in the 21st century, citizens and workers must be able to exhibit a range of functional and critical thinking skills related to information, media and technology.

Information Literacy: Students are able to access and evaluate information effectively and use and manage that information purposefully for the issue or problem at hand. They exhibit awareness and application of knowledge on the ethical/legal issues surrounding the use of information.

Media Literacy: Understanding how, why, and for what purposes, media messages are constructed, and how individuals interpret messages differently. Students create media utilizing the most appropriate media creation tools, characteristics and conventions in diverse, multi-cultural environments.

Information, Communications, and Technology Literacy: Students use digital technologies to manage, integrate, evaluate and create information, and to apply technology effectively, using it as a tool to research, organize, evaluate and communicate.

Life & Career Skills

Today's life and work environments require far more than thinking skills and content knowledge alone. The ability to navigate the complex life and work environments in the globally competitive information age requires students to pay rigorous attention to developing the following life and career skills.

Flexibility and Adaptability: Expertise in adapting to change in varied roles, jobs responsibilities, schedules and contexts and work effectively in a climate of ambiguity and changing priorities, exhibiting flexibility when negotiating and balancing diverse views and beliefs to reach workable solutions.

Initiative and Self-Direction: The capability to set and manage goals and time, and work independently. The capacity to be a self-directed learner going beyond basic mastery of skills and/or curriculum to explore and expand one's own learning and opportunities.

Social and Cross-Cultural Skills: Interacting effectively with others in a respectable, professional manner, and leveraging social and cultural differences to create new ideas and increase innovation and work quality.

Productivity and Accountability: The ability to set and meet goals, in the face of obstacles and competing pressures, to achieve intended result. Demonstrate additional attributes of high quality work, including collaborating and cooperating effectively and respectfully with diverse teams, multi-tasking, and being accountable for results.

Leadership: Utilizing one's own influence and problem-solving skills to work with others towards a shared goal, inspiring others and leveraging their strengths, while demonstrating integrity and ethical behavior.

The value of math education can be found not only in its ability to help contribute to students' college and career readiness, it can also help develop individuals as thought leaders who can understand the world better because of their mathematics capabilities. Mathematics is a common language that can help students unlock complex problems and a lens of understanding by which to make applied and important connections to other fields, professions and disciplines. This project has been developed in order to give educational leaders in and out of the classroom additional tools to help more students recognize this common language so that all students develop skills to lead in the 21st century.

P21 thanks the Mathematical Association of America and the National Council of Teachers of Mathematics for their assistance in developing this Map.



Interdisciplinary Themes

Mathematics as a discipline offers its own unique set of knowledge, skills, and processes.

It also offers the opportunity through an exploration of key math concepts to provide links from school-based learning to interdisciplinary themes that are essential to every student's ability to thrive as a global citizen. Math offers students a lens through which to distinctively view the world, and empowers students with tools for meaningful participation in our democracy and economy. Students are able to discover ways to solve old problems and develop new ways of thinking about the world around them – a skill that is essential to tackling the biggest challenges in our interconnected, global world.

Global Awareness. Mathematics provides opportunities and experiences for students to understand global issues; to work collaboratively with individuals representing diverse cultures, religions and lifestyles in a spirit of mutual respect; and to understand other nations and cultures, including those that shape their thoughts in other languages. The awareness and empathy that comes from solving globally relevant math problems can enhance the ability of students to deepen their content knowledge and global mindfulness simultaneously.



Financial, Economic, Business and Entrepreneurial Literacy. Students must know how to make appropriate personal economic choices as responsible citizens, both at school and at home. These choices require certain mastery skills that will remain pertinent for students as they enter college, pursue careers that require extensive financial, economic and mathematics expertise, and manage every-day financial responsibilities.



Civic Literacy. Study in the field of mathematics provides a context for exploring the rights and obligations of citizenship at the local, state, national and international level, as well as the implications of ethical and civic decisions. Mathematics provide an opportunity for students to become aware of the interplay between governing laws, mathematics problems, and public policy challenges locally, nationally and globally.

Health Literacy. Use of math to calculate nutritional content and assess physical and mental practices can help students develop new tools for monitoring and improving their overall health, as well as understanding habits of healthy behavior.



Environmental Literacy. Students with mathematics skills can become stewards of the Earth when they utilize methods of measuring their impact on the planet and hone their ability to improve environmental conditions. Students may explore environmental issues through math simulations that explore environmental challenges locally, nationally and globally.



Outcomes for P21 Math Skills Map

Creativity and Innovation

- Students compare different ways of approaching traditional mathematical problems and find innovative solutions, using practical examples where appropriate.
- Students listen to and evaluate others' reasoning and offer improvements and corrections, with supporting arguments. They listen to others' feedback and modify their own arguments as needed. They learn from mistakes, and make repeated attempts at solving problems.
- Students look for patterns that suggest creative shortcuts or simplifying frames of reference. They make generalizations from patterns they observe in repeated calculations.
- By discovering fresh insights and communicating them to others, students come to understand that mathematics is a creative endeavor that builds on previous knowledge.

Critical Thinking and Problem Solving

- Students look for a logical structure in addressing mathematical challenges. They are able to make complex choices and construct viable arguments to defend their choices.
- Students identify and ask significant questions about mathematics and engage in analyzing each other's answers.
- Students make sense of applied mathematical problems through analysis and synthesis of evidence, and persevere in solving problems.
- Students analyze how parts of a whole interact with each other in mathematical systems.

Communication & Collaboration

- Students articulate mathematical thoughts and ideas using oral and written communication skills. Using abstract and quantitative reasoning with attention to precision, they construct viable arguments and analyze others' reasoning.
- Students listen effectively to the reasoning of peers. They rephrase another student's explanation or engage in questioning in order to decipher a peer's solution to a mathematical problem.

- Students work efficiently and respectfully in diverse teams, articulating mathematical thoughts and ideas effectively. They use oral, written and nonverbal communication skills, demonstrating how mathematics is used to model problems of broad interest to society.

Information Literacy

- Students identify sources of data, access data, critically evaluate it, and then use it to explore significant questions about our world.
- Students explore new areas of mathematics and its applications, and share what they have learned with others.
- Students learn about mathematics from reliable websites and share their knowledge with others.

Media Literacy

- Students understand how statistics, probabilities, and media messages are constructed for social purposes and how individuals interpret messages differently. They examine the purposes of statistical messages, the tools, characteristics, and conventions used, and how media can influence beliefs and behaviors.
- Students gain a fundamental understanding of the legal and ethical issues surrounding the access, use, and potential distortion of mathematical information, terms, and concepts. They recognize the common tendency to treat quantitative data as truth and to infer patterns where none exist.
- Students present statistical information in ways that support a particular view or help others understand the information.

Information, Communications, and Technology (ICT) Literacy

- Students use tools such as graphing calculators, spreadsheets, computer graphing, computer algebra systems, GPS devices, and online resources appropriately and strategically.
- Students use technology to communicate mathematical insights by constructing appropriate graphical representations of functions and of data.

Flexibility and Adaptability

- Students work in pairs and small groups to address mathematical challenges that involve varied roles and responsibilities, and require working effectively in a climate of ambiguity and changing priorities.

Initiative and Self-Direction

- Students monitor, define, prioritize, and complete tasks independently while balancing tactical and strategic goals to solve mathematical problems.
- Students reflect critically on past experiences solving mathematical problems and connections among mathematical representations in order to inform future problem solving endeavors.

Social and Cross-Cultural Skills

- Students learn about the use of mathematics in other cultures. They recognize the contributions to mathematics from a variety of cultures and the practical needs that led to those contributions.
- Students apply tools of mathematics, such as statistical analysis, to understanding cross-cultural problems and issues.

Productivity and Accountability

- Students set goals, establish priorities and schedules, and meet goals to complete a project.

Leadership and Responsibility

- Students use interpersonal and problem-solving skills to leverage strengths of peers and solve mathematical problems important to their community.
- Students consider the ethical implications of mathematically-based decisions.



Creativity and Innovation

Alignment with Common Core State Standards:

As noted in the [P21 Common Core Toolkit](#), Creativity is not addressed explicitly in the Mathematics Practice section of the CCSS. Creativity is instead strongly implied, especially with regard (but not limited) to topics that involve measurement and data, algebraic thinking, geometry, statistics and probability, and modeling. In addition, the creativity and innovation examples below provide students with opportunities to develop expertise in the following CCSS mathematical practices:

- *Make sense of problems and persevere in solving them*
- *Construct viable arguments and critique the reasoning of others*
- *Model with mathematics*
- *Look for and make use of logical structure*



4th Grade

OUTCOME: Students compare different ways of approaching traditional mathematical problems and find innovative solutions, using practical examples where appropriate.

EXAMPLE: The teacher presents the class with the problem: Five friends are planning a pizza party. If each of them will eat 3 pizza slices and each pizza is cut into 8 slices, how many pizzas will be needed for the party?

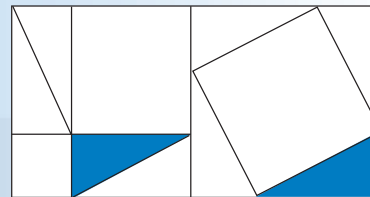
Students work in small groups:

1. One group looks at how many pieces of pizza will be needed. They reason that 5 people eating 3 slices each will require $5 \times 3 = 15$ slices. The group then examines how many pizzas will be needed to get 15 slices.
2. Another group approaches the problem by asking: If each person eats $\frac{3}{8}$ of a pizza, what fraction will 5 people eat?
3. A third group creates paper “pizzas”: Each student draws a large circle and draws 4 lines through the center to create 8 equal-size wedges, or “pizza slices.” Students sort the wedges into piles of 3. They see that 5 piles of 3 wedges each provide a total of 15 “pizza slices.” Since each slice is $\frac{1}{8}$ of a pizza, they reason that 15 slices is $15 \times \frac{1}{8}$, or $\frac{15}{8}$ of a pizza. They find that $\frac{15}{8} = 1 \frac{7}{8}$, which is just under 2 pizzas.



8th Grade

EXAMPLE: Students are shown two different approaches to proving the Pythagorean theorem, such as dissecting a square in different ways, dropping a perpendicular to the hypotenuse from the opposite vertex, or arguing using similar triangles. Students use the Internet to research other proofs of the theorem. Each student writes a brief report comparing two different proofs and presents it to the group.



12th Grade

EXAMPLE: Students learn about Hot Potato, a lottery game that was played in Wisconsin, which cost a dollar to play and had payoff probabilities as shown in the table below. Students then experiment with different ways to model the game. First they estimate how many people out of 10 win each payoff, how many out of 100, how many out of 1000, and so on. Then they model the problem with a probability distribution. They discover that a key payoff value is missing from the probability table: the expected payoff per game. Students then calculate the missing payoff value and learn that the expected payoff per game is about \$0.55. They evaluate the impact of this payoff value from the point of view of the individual player as well as the Wisconsin Lottery Commission.

Payoff	Probability
1	1/9
2	1/13
3	1/43
6	1/94
9	1/150
18	1/300
50	1/2050
100	1/144000
300	1/180000
900	1/270000



Creativity and Innovation (continued)

A 'creativity progression' is a process worth exploring that can expand students' content knowledge and reinforce the application of creativity and innovation skills:

1. Solve exercises (standard solutions)
2. Solve problems (standard solutions)
3. Solve problems using non-standard solutions (creative stretch)
4. Solve a class of problems (metacognitive stretch) using both standard and non-standard solutions
5. Create new problems, and solve using both standard and non-standard solutions
6. Create new classes of problems, and solve using both standard and non-standard solutions

4th Grade



8th Grade



12th Grade



OUTCOME: Students listen to and evaluate others' reasoning and offer improvements and corrections, with supporting arguments. They listen to others' feedback and modify their own arguments as needed. They learn from mistakes, and make repeated attempts at solving problems.

EXAMPLE: Students learn about cicadas (<http://animals.nationalgeographic.com/animals/bugs/cicada/>). They discover that the Magicicada in eastern North America are the world's longest-living cicadas (www.magicicada.org) and that Magicicadas live in the ground and emerge only after 13 or 17 years, depending on the species.

Students explore why these cycles are based on prime numbers. They learn that certain cicada predators might reach a population peak every 2, 3, 4, 5, or 6 years and that if a bird or other predator reaches a population peak during the same year that the cicadas emerge, more cicadas get eaten.

Students look at how many years it would take for 17-year cicadas to emerge in sync with a 5-year predator ($5 \times 17 = 85$ years). They see that a 10-year cicada, however, could be attacked by a 5-year predator every 10 years.

Then students divide into two teams: the cicadas and the predators. The predators pick a number between 2 and 10 (years) to represent how often they will attack the cicadas. The cicadas pick a number to determine how often they will emerge from the ground to breed, which will also make them a target for predators. Then team members confer on possible strategies and experiment by choosing various combinations of predator/cicada numbers. They discover that the predators are like factors and the most successful cicadas have prime-numbered cycles.



EXAMPLE: Students explore different ways of tiling a floor using triangular tiles of a single size and shape. Using geometry software or triangles cut from paper, they experiment with rotations, reflections, and translations of a given triangle. Students compare their tilings with one another, suggest ways to prove that any given triangle can be arranged to cover the floor with no gaps, and respond to each others' propositions. In this process, students examine the relationship between angles formed by a transversal to two parallel lines. They see the connection between this discovery and the fact that the sum of the angles in a triangle is always a straight angle (180 degrees).

EXAMPLE: Students explore the mathematics of the popular puzzle video game Angry Birds (<http://forum.davidson.edu/mathmovement/2011/11/13/algebra-of-angry-birds/>). Working in teams, students build a hands-on version of the game. They create targets by stacking blocks or other lightweight objects, and they construct a toy catapult or slingshot to aim at the target (or toss a beanbag). After experimenting by trying to knock down the target from various starting points, students come up with different equations for parabolas to see which will work to knock down the structure from a given distance.



Creativity and Innovation (continued)

4th Grade



8th Grade



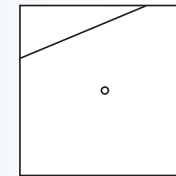
12th Grade



OUTCOME: Students look for patterns that suggest creative shortcuts or simplifying frames of reference. They make generalizations from patterns they observe in repeated calculations. CCSS alignment: Look for and express regularity in repeated reasoning.

EXAMPLE: Students skip count from a variety of different starting numbers. They identify patterns and explain the patterns they observe. For example, when skip counting by 5, they see that the last digit repeats every second time; when skip counting by 7, the last digit repeats every 10 times. Students create skip counting patterns with cycles of different lengths. They also identify skip counting patterns in which the last digit repeats at varying intervals. They investigate and record what sorts of cycle lengths and patterns are possible.

EXAMPLE: Each student is given a square piece of cardboard with a hole in the middle and a drawn line segment connecting two sides of the square. The line segment may connect any two sides at any point as long as it does not pass through the middle of the square. Students spin the square on the tip of a pencil and observe the optical illusion of a circle created by the drawn line segment. Students come up with a conjecture to identify where the illusionary “radius” is. Then they translate their square to a Cartesian coordinate plane and discuss the curve tangent to the locus of the line segments as they rotate about the center.



EXAMPLE: Students use a computer algebra system to factor completely the polynomial $x^n - 1$ for positive integer values of n . They conjecture a relationship between n and the number of factors and post their conjecture on a class website. Students review each others' conjectures, and then each student revises or refines his or her own conjecture.



Creativity and Innovation (continued)

4th Grade



8th Grade



12th Grade



OUTCOME: By discovering fresh insights and communicating them to others, students come to understand that mathematics is a creative endeavor that builds on previous knowledge.

EXAMPLE: Students explore Mayan mathematics (www.hanksville.org/yucatan/mayamath.html). They compare and contrast how numbers are represented in our Arabic system and in the Mayan system. They practice writing Mayan numerals and create addition and subtraction problems in the Mayan system for one another to solve (<http://mathforum.org/k12/mayan.math/index.html>).

EXAMPLE: Working in small groups, each team of students explores a different numeral system and prepares a report for the rest of the class explaining how numbers are represented in the system, how to do arithmetic operations, and the significance of the system. For example, binary (base 2) is the basis for electronic computing; duodecimal (base 12) is more convenient for adding fractions than our decimal (base 10) system is; and sexagesimal (base 60) was used by the ancient Sumerians and Babylonians.

EXAMPLE: Working in teams, students study examples of Kolams from Southern India or Sona from Angola and Zaire and explore the mathematics in these complex geometric art forms. They explore Kolam designs based on Fibonacci numbers (<http://vindhiya.com/Naranan/Fibonacci-Kolams/>) and examine how an array of dots can give rise to a one-line sona (www.beloit.edu/computerscience/faculty/chavey/sona/). Using the concepts they have learned, students try to create their own Kolams or Sona.



Critical Thinking and Problem Solving

Common Core State Standards Alignment:

As the [P21 Common Core Toolkit](#) points out, Critical Thinking and Problem Solving skills most explicitly align with the following Common Core State Standards' mathematical practices:

- *Make sense of problems and persevere in solving them*
- *Reason abstractly and quantitatively*
- *Model with mathematics*
- *Look for and make use of structure*

4th Grade



OUTCOME: Students look for logical structure in addressing mathematical challenges. They are able to make complex choices and construct viable arguments to defend their choices.

EXAMPLE: Student create coded messages and decipher each others' messages. They learn about "Caesar ciphers" used by Julius Caesar to send secret messages to his military troops—similar to what we call "substitution ciphers." In this type of cipher, each letter is replaced by a different letter. For example, if each letter were replaced by the letter four places down in the alphabet, then the sentence, "Meet at the gate" would be encoded, "Qiix ex xli kexi."

Working in small groups, each team creates a coded message using a substitution cipher and then gives its message to another group to decipher.

Students learn that Morse Code and Braille are also like secret codes and that computers are "programmed" with coded messages.

8th Grade



EXAMPLE: Students examine a local building that has stairs but no ramp for wheelchairs. Working in groups, students identify the best place to install a ramp. Then they determine the appropriate slope, decide whether or not the ramp should have a switchback, and design the ramp using the Pythagorean Theorem.

12th Grade



EXAMPLE: Students go online to research current interest rates for unsubsidized Stafford loans, home equity loans, and home equity lines of credit. Then they devise a plan for borrowing \$5,000 per year to help finance four years of college while keeping costs as low as possible:

Students calculate monthly payments and the total amount that would be owed (1) at graduation, (2) five years after graduation, and (3) 10 years after graduation.

Students also determine how the strategy would change if one of the options were a federally subsidized Stafford loan.

For both subsidized and unsubsidized Stafford loans, students need to consider the limits on how much may be borrowed each year.



Critical Thinking and Problem Solving (continued)

MATH



4th Grade



8th Grade

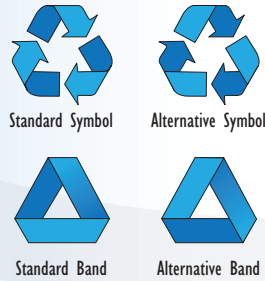


12th Grade

OUTCOME: Students identify and ask significant questions about mathematics and engage in analyzing each others' answers.

EXAMPLE: Each student creates a Möbius band by starting with a long, thin strip of paper, giving one end a half-twist, and taping the two ends together. Students draw a line along the center of their band and discover that the line ends where it began, confirming that the band has only one side.

Students view pictures of the standard recycling symbol and recognize that it represents a Möbius band. The class discusses why this one-sided figure seems appropriate for use as a recycling symbol. Then students look for recycling symbols in the community. By studying the symbols carefully, they may discover that sometimes the standard symbol appears, but sometimes a slightly different symbol is used.



Students try to determine how the alternative symbol differs from the Möbius band symbol. Then they create the alternative band from a strip of paper by giving one end three half-twists and then taping the ends together. They debate whether the alternative band is one-sided or two-sided and then draw a line along the center to determine the answer.

EXAMPLE: The class divides into two groups. In one group, students use a piece of string and a ruler to measure the circumference (c) and diameter (d) of circular objects, such as the lid of a jar, the face of a clock, or a pie plate. For each object measured, they calculate c/d . Then they calculate the average of each result to come up with an approximate value for π .

In the other group, students use the method developed by Archimedes, using inscribed and circumscribed polygons.

Students compare the two groups' results. They recognize that π is an irrational number, so it cannot be measured precisely. Then they research how people in different cultures have tried to calculate π from ancient to modern times.

EXAMPLE: Students explore the napkin ring problem: if a hole of height h is drilled through the center of a sphere, the volume of the portion of the sphere that remains does not depend on the size of the original sphere; it depends only on h . They share and critique their insights into why this is so. Then students explore mathematician Keith Devlin's 2008 discussion of the problem at www.maa.org/devlin/devlin_04_08.html, where Devlin provides the full computation and explains why some solutions posted online are incorrect. Students explore solutions currently appearing online and assess which solutions are accurate and which are not.



Critical Thinking and Problem Solving (continued)



4th Grade



8th Grade



12th Grade

OUTCOME: Students make sense of applied mathematical problems through analysis and synthesis of evidence, and persevere in solving problems.

EXAMPLE: Students work in small groups to design a parking lot. Given a flat, rectangular space with set dimensions, each group arranges rows of parking spaces to be easily accessible while allowing for a maximum number of cars to park. Students may visit an actual parking lot to assess and measure parking space width and angles. Each group creates a drawing of its design and presents it to the class.

EXAMPLE: Students choose three meals from a fast food restaurant—breakfast, lunch, and dinner—and obtain nutrition information from the restaurant or from its website. They add up the calories for each meal and compute the total calories that one would consume in a day from the three meals.

To analyze their totals, students refer to the 2010 Dietary Guidelines for Americans (<http://health.gov/dietaryguidelines/dga2010/DietaryGuidelines2010.pdf>). They use the table on page 14 of the Guidelines to estimate their caloric needs based on age, gender, and physical activity level.



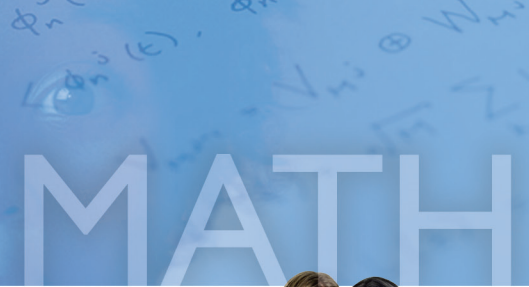
Students also add up the protein, fat, sodium, and carbohydrates for the three meals they selected, list the percentage for each item, and compare it with the Guidelines.

EXAMPLE: Students read about the mathematics of three-dimensional maps that a team of researchers has designed for measuring the environmental value of open space areas with no roads (www.sciencenews.org/view/generic/id/8519/title/Math_Trek_Miles_from_Nowhere).

Students then explore current policies pertaining to conserving roadless areas, such as the controversial “Roadless Rule” and determine how the mathematical maps could be used to improve policies for conserving open space. They craft a letter to their congressional representative or another policymaker explaining their analysis.



Critical Thinking and Problem Solving (continued)



4th Grade

8th Grade

12th Grade

OUTCOME: Students analyze how parts of a whole interact with each other in mathematical systems.

EXAMPLE: Students learn about different calendar systems and come to realize that the movements of the earth, moon, sun, and solar system interact with each other and affect how we mark time.

- The Gregorian (Western) calendar is based on Earth's orbit around the sun. Since one year (365.25 days) is slightly longer than Earth's current orbit, three days are skipped every four centuries.
- The Muslim calendar is based on lunar cycles in which the beginning date for each month is based on when the new moon can be observed. The beginning of the month varies among different Muslim countries, and western Muslim countries are likely to observe the beginning of the month a day earlier than eastern Muslim countries.
- The Chinese calendar is based on 60-year cycles and uses the phases of the moon to determine dates for each New Year.
- The Hebrew calendar uses lunar cycles and adds an extra month every two or three years to adjust to the solar year.



EXAMPLE: Students debate the merits of ranked-choice voting and its recent use in several local elections (<http://io9.com/5857724/ranked-choice-voting-does-a-mathematical-algorithm-make-for-better-elections>). To compare ranked-choice voting with traditional voting systems, the class conducts a mock election with three to five candidates. The candidates may be actual persons, fictional characters, favorite songs, or other choices. Instead of voting for just one candidate, students rank their top three choices.

In tabulating results, students initially count only each voter's first choice. If no candidate receives more than 50 percent of the vote, students take three different approaches for determining the winner:

1. Simply declare the candidate with the most first-choice votes to be the winner.
2. Hold a run-off election between the two candidates who received the most first-choice votes.
3. Tabulate voters' ranked choices: First, the candidate with the lowest number of first-choice votes is eliminated, and the second and third choices of that candidate's supporters are added to the counts of the other candidates. The votes are tabulated again, and candidates are eliminated until one candidate receives a majority of the eligible votes.



Students compare the three outcomes and talk about how the math we choose can affect the kind of government we get. Students may also read mathematician Keith Devlin's discussion of election math, available online at www.maa.org/devlin/devlin_11_04.html.



Communication & Collaboration

Common Core State Standards Alignment:

As noted in the [P21 Common Core Toolkit](#), Communication and Collaboration skills are most explicitly aligned with the following Common Core State Standards' mathematical practices:

- *Construct viable arguments and critique the reasoning of others*
- *Attend to precision*
- *Look for and express regularity in repeated reasoning*

4th Grade



EXAMPLE: Students pair up and play “The Factor Game,” which challenges players to find factors of numbers on a gameboard. In an online version, available at <http://illuminations.nctm.org/activitydetail.aspx?id=12>, the gameboard can include numbers up to 30, 49, or 100. Students can start with the simplest version, try to identify the mathematics that leads to a winning strategy, and then test their strategies using larger numbers.

Students then present and compare their playing strategies with their classmates' approaches to determine optimal strategies. At a “Family Math Night,” pairs of students present the game to their family members and explain strategies for winning.

8th Grade



EXAMPLE: Working in small groups, students compare the cost of buying a hybrid versus a non-hybrid car. Students research pricing and fuel miles-per-gallon estimates for comparable models of hybrids and non-hybrids. They factor in average local gas prices over x years. Using their knowledge of linear functions, students then analyze the overall cost of the hybrid versus the non-hybrid vehicle over x number of years, using assumptions regarding the average price of gas and how many miles the car will be driven each year. Students analyze how changing the number of years or number of miles to be driven would affect the outcome. Each group presents its results and conclusions to the class.



12th Grade



EXAMPLE: As a joint activity with a social studies class, students study the history of how seats have been allocated in the U.S. House of Representatives. They analyze the mathematics behind different plans and underlying socio-political issues, such as the effects on small versus large states or rural versus urban populations. Students then solve various “fair division” problems for their city council or state house of representatives using different plans. Finally, students hold a debate in which students advocate for different plans, considering both the mathematical and social issues that go into the allocation of seats.



Communication & Collaboration (continued)

4th Grade



8th Grade



12th Grade



OUTCOME: Students listen effectively to the reasoning of peers. They rephrase another student's explanation or engage in questioning in order to decipher a peer's solution to a mathematical problem.

EXAMPLE: A soccer team is planning an end-of-season party. There are 4 tables in the party room, and the players have 36 balloons to decorate the tables: 12 red balloons, 12 yellow, and 12 blue. There are 14 players on the soccer team. How many balloons should each table have? What is the best way to divide up the colors? And how many players should sit at each table?

Students work in small groups. In each group, one student presents a plan and explains why it's the best way to distribute the balloons and the seating. A second student repeats the first student's explanation in his or her own words. Then the other students in the group ask about the explanation to help make sure it's a good plan and that it is clear to everyone. Then each group presents its plan to the whole class.

EXAMPLE: Students work in groups to create graphical representations of common, well-known functional relationships. For example, one group might examine the number of people standing in line at the post office at various hours of the day; another group might consider the amount of business at a local coffee shop at various hours; a third group might compare different school lunchroom menus with the number of students who bought lunch that day.

In each group, students select a title for their graph and provide that title to the teacher only. The teacher compiles a list of titles. Students post each group's graph around the classroom, and then each student tries to figure out which title is associated with each graph. Once everyone has matched each graph with a title, students share their reasoning for each match. When varied opinions regarding titles come up, students discuss the qualitative aspects of the function that has been graphed. The discussion also provides avenues for students to question peers regarding their explanation and reasoning.

EXAMPLE: Students play the role of teacher for a short, specific mathematics lesson. In advance of the class, they study the material, plan what supportive materials or manipulatives to use, and develop a lesson plan. Then the students "teach" the class and assign follow-up work to assess what their peers learned. The students in the class are encouraged to pose challenging questions that could possibly stump their student "teacher" and lead to further discussion or investigation of the topics under study.



Communication & Collaboration (continued)

4th Grade



8th Grade



12th Grade



OUTCOME: Students work efficiently and respectfully in diverse teams, articulating mathematical thoughts and ideas effectively. They use oral, written and nonverbal communication skills, demonstrating how mathematics is used to model problems of broad interest to society.

EXAMPLE: Students work in groups to design a bedroom. One student in the group plays the role of client, and the others act as the design team. The design team is given building constraints on floor area, wall area, and minimum number of windows. The design team interviews the client for preferences regarding: window and door placement; size and placement of bed, desk, and closets; and size and locations for any wall posters or other decorative items that the client asks to have included. The design team produces a scale drawing of the room with an explanation of why it satisfies the constraints and the wishes of the client. The client checks the design and sends it back for more work if necessary.

EXAMPLE: Students form investigative teams. Each team is asked to investigate the crime rate in a particular city, represented by the variable x . Each team then formulates a question about a possible causal variable y . For example, a team might ask if crime rates are lower in cities with a larger police force, or higher in cities with higher poverty rates. The team then chooses 30 to 40 other cities with which to compare their city's crime rate. By searching on the Internet, they collect data on x and y . If team members find their data too difficult to access, they consider revising the question. For example, if they are not finding suitable data on poverty rates, team members might decide to investigate the size of the police force in each city instead. Team members analyze the data they have collected, define the relationship between x and y , and discuss questions such as the reliability of the data, its statistical significance, and the validity of the sources. Each team prepares a presentation, explaining the findings and team members' conclusions.



EXAMPLE: Students form teams to engage in a modeling project similar to problems presented in the Mathematical Contest in Modeling run each year by COMAP. For example, the following problem is adapted from the 2009 competition:

Many cities and communities have traffic circles. Some traffic circles are large and have many lanes, such as at the Arc de Triomphe in Paris, and the Victory Monument in Bangkok. Some have only one or two lanes in the circle. Some traffic circles position a stop sign or a yield sign on every incoming road that gives priority to traffic already in the circle. Others post yield signs in the circle at each incoming road to give priority to incoming traffic. Still others have a traffic light where each incoming road meets the circle (with no right turn allowed on a red light). Other designs may also be possible.

Students form teams, and each team develops a model for creating a traffic circle in their own community or in a nearby city (or modifying an existing traffic circle).

1. First, team members discuss how many lanes should be in the circle and the rate of traffic flow. If feasible, they may collect actual traffic data.
2. Team members outline alternatives for the placement of stop signs or yield signs and other design factors. Then they work together
(continued on next page)



Communication & Collaboration (continued)

4th Grade



8th Grade



12th Grade



- to determine how to best control traffic flow in, around, and out of the circle.
3. Team members determine the best alternatives and explain what factors influenced their decision, such as the effect on traffic flow and safety.
 4. Team members refine their model. For example, if traffic lights are recommended, students develop a method for determining how many seconds each light should remain green, which may vary according to the time of day and other factors.



Information Literacy

Common Core State Standards Alignment:

As noted in the [P21 Common Core Toolkit](#), Information Literacy skills are most explicitly aligned with the following Common Core State Standards' mathematical practices:

- Construct viable arguments and critique the reasoning of others
- Attend to precision
- Look for and express regularity in repeated reasoning

4th Grade



8th Grade



12th Grade



OUTCOME: Students identify sources of data, access data, critically evaluate it, and then use it to explore significant questions about our world.

EXAMPLE: Students work in teams using the U.S. Census Bureau “State and County Quick Facts” (<http://quickfacts.census.gov/qfd/index.html>) to look up population facts for five states in different parts of the country. The site shows each state’s population in 2000 and in 2010, and the percent change. Students create a table listing this information for each of the five states and for the total U.S. population. Then students estimate the 2020 population for each of the five states and for the U.S. as a whole. Groups share their tables with the class, and students talk about which areas of the country show the most population growth and possible reasons to explain population growth or decline.



SAMPLE:

State	Population in 2000	Population in 2010	% Change 2000–2010	Forecast for 2020
California				
Nebraska				
Louisiana				
Vermont				
South Carolina				
U.S. Total				

EXAMPLE: Students use the 2010 census website (<http://2010.census.gov/2010census/data/>) to access the 2010 Census Brief, “The Hispanic Population: 2010.” Each student selects a U.S. Hispanic population category identified by country of origin (Mexican, Dominican, Puerto Rican, etc.) and investigates data associated with that population group: the number of residents from the selected country of origin in 2000 and 2010 and the percent change over those 10 years. Students then use the Internet to research the population and average annual income for the selected country of origin. Students look for correlations between immigration numbers and the population and income levels for the country of origin. Students share their results with the class. In a group discussion, students analyze the influence of income levels in Latin American countries on rates of emigration to the United States.



EXAMPLE: Students use the Department of Labor’s Bureau of Labor Statistics (<http://www.bls.gov/>) to investigate regional U.S. employment data. Each student focuses on two states in different parts of the country. For each state, the student constructs: (1) a graph showing unemployment rate trends over the past 10 years; (2) a graph comparing unemployment trends within three different job sectors; and (3) a graph or table comparing unemployment trends in the state as a whole with trends in a particular city within the state. Based on the data collected, students identify a good place to relocate and look for a job. Students present their findings to the class and share their thoughts about the extent to which data might influence their own future decisions about where to live or what sort of career to pursue.



Information Literacy (continued)

4th Grade



8th Grade



12th Grade



OUTCOME: Students explore new areas of mathematics and its applications, and share what they have learned with others.

EXAMPLE: Students explore the topic of fractals on an age-appropriate website, such as <http://math.rice.edu/~lanius/frac/>, which explains what fractals are, why we study them, how people use them to solve real-world problems, and how to make various types of fractals. Using information found online, each student creates a fractal, shares it with the class, and explains how it works.

EXAMPLE: Students analyze a set of sports statistics, such as baseball box scores, college basketball data, tennis rankings, or college football standings. They examine how winners are chosen and explore possible alternatives for ranking players or teams in ways that are accurate and fair.

EXAMPLE: Students explore knot theory and its role in genetics (www.oglethorpe.edu/faculty/~j_nardo/knots/history.htm). They learn how the DNA in every cell of our bodies is made up of long, coiled strands that get mathematically knotted in ways that lead to chemical changes in the DNA. Using string, rope, or even tangled audio headphone chords, students create several types of knots, such as those shown at www.mathedpage.org/pcmi/knot-theory.pdf. Then students examine each other's knots and identify them by type: unknot, trefoil, figure eight, five-star, one of the six's, etc. Students photograph and label the knots and display them by creating a poster or posting them to a photo gallery on a class website.



Information Literacy (continued)

4th Grade



8th Grade



12th Grade



OUTCOME: Students learn about mathematics from reliable websites and share their knowledge with others.

EXAMPLE: Students investigate the topic of mathematical tessellation by reading articles about it. Then they experiment with ways of tiling a floor or creating a geometric pattern for a quilt by building a tessellation using an interactive online tool such as:

NCTM's Illuminations website (<http://illuminations.nctm.org/ActivitySearch.aspx>), or the National Library of Virtual Manipulatives (<http://nlvm.usu.edu/en/nav/vlibrary.html>). Students then work in small groups to share their results. The group identifies shapes that fit together to form a tessellation and shapes that do not.

EXAMPLE: Students work in groups to investigate a field of modern mathematics research, such as the "four-color map theorem," fractals, buckyballs, or the mathematics of DNA structure. Each group presents its findings to the class.

EXAMPLE: Students work in groups to investigate the history of a topic within the class's current field of study, using websites such as the MacTutor History of Mathematics (<http://www-history.mcs.st-and.ac.uk/>). For example, if the class is studying algebra, each group explores the history of an algebra topic, such as quadratic equations, group theory, or set theory. Each group creates a wiki entry based on its findings.



Media Literacy

Common Core State Standards Alignment:

As noted in the [P21 Common Core Toolkit](#), Media Literacy skills are most explicitly aligned with the Common Core State Standards' mathematical practice calling for students to use appropriate tools in a strategic way. In addition, the examples below provide students with an opportunity to develop expertise in the following Common Core State Standards' mathematical practice:

- *Construct viable arguments and critique the reasoning of others*
- *Model with mathematics*

4th Grade



8th Grade



12th Grade



OUTCOME: Students understand how statistics, probabilities, and media messages are constructed for social purposes and how individuals interpret messages differently. They examine purposes of statistical messages, the tools, characteristics, and conventions used, and how media can influence beliefs and behaviors.

EXAMPLE: Students make a list of all the media messages they encounter one morning before school. For example, they might see a health claim on a cereal box, hear news headlines on TV or radio, spot an ad on the side of a local bus, and notice posters for a local election campaign. Then they use a spreadsheet to categorize the purposes of each message, the tools used, and other characteristics. For example, a student might decide to track which messages are food-related, which ones are health-related, which ones are trying to get them to buy something, which ones use electronic media, and which messages are aimed at children. Students tally the number of messages in each category.

EXAMPLE: Students study how the U.S. Bureau of Labor Statistics (BLS) uses the Consumer Price Index (CPI) to measure inflation. Then students examine how the European Union measures inflation using the Harmonized Index of Consumer Prices. After comparing the two approaches, students read media articles about the CPI and inflation and discuss how the government's way of measuring the CPI affects public policy decisions.



EXAMPLE: Students read articles about the Centers for Disease Control (CDC) 2004 report claiming obesity was responsible for 400,000 deaths each year and the ensuing outcry leading to the reduction of this figure in the following year's report to 24,000. They discuss how correlations are formed and the importance of confounding factors related to spurious relationships.



Media Literacy (continued)



4th Grade



8th Grade



12th Grade

OUTCOME: Students gain a fundamental understanding of the legal and ethical issues surrounding the access, use, and potential distortion of mathematical information, terms, and concepts. They recognize the common tendency to treat quantitative data as truth and to infer patterns where none exist.

EXAMPLE: Students examine the numbers on food labels. Each student lists the serving size on a bag of chips, the amount of fat per serving, and the “daily value” percentage. Then the student estimates the number of servings he or she would actually eat, and recalculates fat per serving and “daily value” based on the number of servings (see sample below). Students do a similar estimation and calculation for cereal, cookies, salad dressing, milk, or other foods. Students can also calculate estimates for saturated fat, sugar, or sodium. Students talk about their findings and discuss the fact that (as footnoted on food labels) “daily value” is based on a 2,000-calorie-per-day diet—and some people need more than 2,000 calories per day while others need less.



SAMPLE:

Item	Service size	Fat per serving	Daily value	Estimated # of servings	Estimated fat	Estimated daily value
Tortilla chips	7 chips	7 grams	10%	2 servings (14 chips)	14 grams	20%
Salad dressing	2 Tbsp	11 grams	17%	1/2 servings (1 Tbsp)	5.5 grams	8.5%

EXAMPLE: Student teams create questions for a student survey to determine whether or not to introduce “Meatless Mondays” in the school cafeteria. Each team creates survey questions and polls 30 students in the school. Each team then compiles descriptive statistics on the findings such as gender and grade levels of the students polled, the percentage of students who like to eat meat every day, percentage who never eat meat, percentage who sometimes eat meat, and comments for and against “Meatless Mondays.” Students then compare the different teams’ findings and discuss margin of error in relation to the findings and the role of random representative samples.



EXAMPLE: Students read articles about the so-called “Climategate” scandal of 2009, in which critics charged that some scientists had manipulated data to overstate global warming trends. They also read about more recent studies confirming or refuting earlier data on the impact of global warming. Based on an analysis of relevant data and charts, students break into groups to argue both sides of this issue to explain global warming and how it relates to climate change.



Media Literacy (continued)

4th Grade



8th Grade



12th Grade



OUTCOME: Students present statistical information in ways that support a particular view or help others understand information.

EXAMPLE: The entire class creates a list of state(s) or countries where each student's parents were born. Students then work in small groups to present the information in a way that will make it easier to understand. For example, one group might create a table listing the total number of parents from Texas, the total number from other U.S. states, the number from Mexico, etc. Another group might create a chart using different colors of dots to show how many parents came from each state or country, and students might experiment with different ways of arranging the dots. Another group might begin with a map and mark each state or country with stick figures or other symbols to show how many parents were born there.



EXAMPLE: Students create a class list showing each student's birthday month and shoe size. Working in small groups, students create a scatter plot or chart depicting the relationship between birthday months and shoe sizes. Each group analyses the information and identifies a correlation between the two sets of data, such as, "Classmates born in winter have bigger feet than those born in summer," or "Most fourth graders born in August wear size 7 shoes." Each group prepares a presentation to convince the rest of the class of the statistical merit of the correlation the group has identified. Classmates prepare oral or written rebuttals arguing why the correlation may be based on insufficient evidence.

EXAMPLE: Students research statistics about the population of a particular country, including the total population, the number of people in each ethnic group, population percentages associated with particular religions, the total workforce, and the percentage of workers in specific sectors, such as agriculture and government service. Students analyze the data and write a "country fact sheet" that aims to attract visitors to the country they have studied. They create a chart, graph, or table to present the data discussed in their write-up. The fact sheet may also include attractive photos or other images from the country.



EXAMPLE: Students work in small groups, and each group researches a country in the developing world. Each student within a group researches economic development in a particular region of the group's assigned country and collects statistics about that region, such as per capita income, access to safe water, number of cell phones, infant mortality rate, etc. Each member of the group creates a table based on the information he or she has collected, and then the group "publishes" its tables online and writes a short report comparing and contrasting data from the country's different regions.



Information, Communications, and Technology (ICT) Literacy

Common Core State Standards Alignment:

As noted in the [P21 Common Core Toolkit](#), ICT Literacy skills are most explicitly aligned with the following Common Core State Standards' mathematical practice:

- *Use appropriate tools strategically*

4th Grade



OUTCOME: Students use tools such as graphing calculators, spreadsheets, computer graphing, computer algebra systems, GPS devices, and online resources appropriately and strategically.

EXAMPLE: Students list their daily activities and note how many hours per day they spend on each activity. For example, a student might spend 8 hours sleeping, 7 hours at school, 1 hour traveling to and from school, 1 hour doing homework, 1 hour doing an arts or sports activity, 1 hour reading, 2 hours eating, 1 hour playing outside, 1 hour watching TV and 1 hour playing games. The total should add up to 24 hours. Then, students use the following internet tools to create visual representations:

- National Library of Virtual Manipulatives: (<http://nlvm.usu.edu/en/nav/vlibrary.html>) to create a bar chart and a pie chart with percentages
- Shodor's Project Interactivate (www.shodor.org/interactivate/) to create a circle graph and a bar graph
- NCTM's Illuminations website (<http://illuminations.nctm.org/ActivitySearch.aspx>) to create a data graph.

8th Grade



EXAMPLE: Students access current stock market data online showing the percentage increase or decrease for each company in the New York Stock Exchange. They generate three random samples showing percentage changes for 5 companies, 15 companies, and 30 companies. Using a graphing calculator or similar tool, students explore different ways to (1) show the data graphically for each sample; and (2) summarize the data for each sample using a single number. Students compare the results for the different sizes of random samples and discuss why one might analyze stock market changes using a random sample instead of collecting data from every company on the New York Stock Exchange.



12th Grade



EXAMPLE: Student teams research current interest rates and loan rates, and then compare different ways someone with \$500 per month of disposable income could purchase a \$10,000 automobile. For each of the following payment methods, students generalize patterns from a few payment cycles and use a spreadsheet to study how long it will take to pay off a balance, the total of all payments made, and the total amount of interest charged over the life of the balance:



1. Using a credit card for the entire purchase and paying only the minimum balance each month.
2. Using a credit card for the entire purchase and paying \$500 per month.
3. Taking advantage of a special loan from the automobile dealer: 1% interest for the first six months and 5% thereafter.
4. Delaying the purchase in order to save up for it first: Spending \$100 per month on public transportation and putting \$400 per month into a savings, checking, or money market account for one year at current interest rates. Then, using the amount saved as a down payment for the car, using a credit card for the remainder due, and spending \$500 per month to pay off the credit card loan as soon as possible.



Information, Communications, and Technology (ICT) Literacy (continued)

4th Grade



8th Grade



12th Grade



OUTCOME: Students use technology to communicate mathematical insights by constructing appropriate graphical representations of functions and of data.

EXAMPLE: Students are given a pretend \$20 to spend and a list of foods for purchase. They look up prices using an online grocery store or other source, trying to make wise choices with their limited funds. They record the name, cost, quantity and total amount they would spend for each item. Using online tools, they create a pie chart comparing the amount to be spent on breakfast, lunch, dinner, and snacks. They describe each portion as a fraction and as a percentage and discuss the relationship between these two representations of rational numbers. They also convert the pie chart to a graph and compare these two ways of depicting the data.



EXAMPLE: Students use statistical software such as Tinkerplots or Fathom to compare the variables associated with different passenger vehicles. Students identify miles-per-gallon and price for each vehicle. They create a scatter plot to show the relationship between the two variables for the different vehicles. Then students look for patterns that might indicate a relationship between the two variables and create an equation to model the behavior of the relationship.



EXAMPLE: Students use graphing calculators, computer algebra systems, or both, to assist in examining patterns in population growth or decline of a particular animal species using data from the U.S. Fish & Wildlife Service or another source. For example, students might study the changes in the population of Canada geese by building functions and examining graphs to answer problems regarding long-term trends. Students can share their findings with a school biology class by developing a lesson on the use of mathematics in examining biological developments.



Flexibility and Adaptability

Common Core State Standards Alignment:

As the [P21 Common Core Toolkit](#) states, Life and Career skills such as Flexibility and Adaptability, align well with the Common Core State Standards' mathematical practice standard which calls for students to make sense of and persevere in solving problems. The examples below provide students with opportunities to develop expertise in the following Common Core State Standards' mathematical practices:

- *Construct viable arguments and critique the reasoning of others*
- *Model with mathematics*

4th Grade



EXAMPLE: In small groups, students plan a school purchase of items such as furniture, school supplies, and playground equipment based on a fixed budget. Students determine the prices of items they are considering and discuss the tradeoffs required to stay within budget. Group members make individual recommendations, negotiate, and come to a group decision. Members of the group can also take on various roles such as scheduler, group moderator, accountant, record keeper, writer, and speaker to present the chosen purchase order to the class.



8th Grade



EXAMPLE: Working in small groups or pairs, students compare the amount of fresh water needed to produce various animal and vegetable foods, using online sources such as the following:

- www.waterfootprint.org
- www.agrometeorology.org/topics/new-information-for-agrometeorologists/globalization-of-water-the-connection-between-water-consumption-and-production-in-a-globalizing-world.-water-footprints

Then, each group creates a one-week menu (five lunches) for the school cafeteria that aims to minimize water use while being both appealing and nutritious for students. Students post their proposed menus in a prominent location at the school.



12th Grade



EXAMPLE: Students work in pairs or small groups to collect four news reports on the results of public opinion polls about the debt ceiling. Each pair or group lists the survey methods of the polls discussed in the news reports. For example, methods may include the number of respondents in a random sample; the age and/or gender mix of persons polled; margin of sampling error; languages used for polling; number of landline respondents; number of cell phone respondents; and factors used in weighting each sample, such as education, race, ethnicity, size of household, and region. Group members then compare the results of each poll by examining the polling dates and other factors. A member from each group shares the group's analyses with the class in a short presentation. Then, each student identifies two or three polls that he or she believes to be most accurate, writes a short paragraph to explain why, and posts it on a class message board.



Initiative and Self-Direction

Common Core State Standards Alignment:

As the [P21 Common Core Toolkit](#) states, Life and Career skills such as Initiative and Self-direction align well with the Common Core State Standards' mathematical practice standard calling for students to make sense of and persevere in solving problems. The examples below also provide students with opportunities to develop expertise in the following Common Core State Standards' mathematical practices:

- *Construct viable arguments and critique the reasoning of others*
- *Look for and make use of structure*
- *Look for and express regularity in repeated reasoning*

4th Grade

OUTCOME: Students monitor, define, prioritize, and complete tasks independently, while balancing tactical and strategic goals to solve mathematical problems.

EXAMPLE: Students work alone or in small groups to look for a method of finding the sum of all whole numbers from 1 to 100 without adding up all the numbers individually. Students who need help getting started might first wish to examine ways of adding up the numbers from 1 to 10.

8th Grade

EXAMPLE: Students work alone or in small groups to tackle the non-standard mathematical problem, "Two primes make one square" (<http://nrich.maths.org/5743>). First, students identify prime numbers between 1 and 100 that are the sum of two square numbers (for example, $4=2+2$; $9=2+7$; and $16=5+11$). Then, students try to find square numbers that are not the sum of two primes. By identifying prime numbers and listing the squares of numbers from 4 to 20, students may discover that 121 and 289 cannot be expressed as the sum of two square numbers.

Students may also discover that whenever an odd square number is the sum of two primes, one of the primes must be the number 2. Students are asked why one of the primes must be 2.

12th Grade

EXAMPLE: Early in the school year, each small group of students is given a pretend \$1,000 to invest. Team members compare risk and return rates on various stocks, bonds, mutual funds, certificates of deposit, and other investments. They research financial forecasts, inflation rates, compounding of reinvested earnings, and the impact of taxes in selecting their investments. Each group shares its investment strategy with the class, and students track their investments over time by calculating what the return on their original \$1,000 would be each month. They reinvest each month's earnings and determine whether to change investment vehicles or keep the money where it already is. After six months, each group calculates its total return, reviews its strategy, and presents a report to the class. Students discuss which groups' strategies were most effective and why.



Initiative and Self-Direction (continued)

4th Grade



8th Grade



12th Grade



OUTCOME: Students reflect critically on past experiences solving mathematical problems and make connections among mathematical representations in order to inform future problem solving endeavors.

EXAMPLE: Students use their understanding of models for multiplication (equal-sized groups, arrays, area models), place value, and properties of operations (especially the distributive property) to develop efficient, accurate methods of computing the products of multi-digit whole numbers. Then, students work in small groups to discuss their new methods and make sure that each method can be generalized to apply to all whole numbers.

EXAMPLE: Students compare a cell phone plan costing \$29.99 per month plus \$0.10 per text message with a plan costing \$39.99 and \$0.02 per text message by calculating how many text messages would need to be sent and received each month in order for the second plan to be less expensive than the first. Students then create an algebraic formula for computing the total cost of any cell phone plan based on per month and text message costs. Next, students draw connections among tables, graphs, and algebraic solutions for such problems. They recognize how such connections can form a basis for solving future linear and non-linear problems in multiple ways.



EXAMPLE: Students review the algebraic formula for exponential growth and use the formula to solve a basic problem involving the spread of a viral disease: If 100 people are currently infected, and the number of people infected doubles every twelve hours, how many people will be infected after one week?

The class discusses the exponential spread of infections and identifies other relevant examples of exponential growth, such as: bacteria growth; human population growth; the growth of atmospheric carbon dioxide; and, in finance, the compounding of interest. Then students work in groups: each group poses a question involving exponential growth and gives it to another group to answer. Each group shares its results with the class.



Social and Cross-Cultural Skills

Common Core State Standards Alignment:

As the [P21 Common Core Toolkit](#) states, Life and Career skills such as Social and Cross-cultural Skills align well with the Common Core State Standards' mathematical practice standard calling for students to make sense of and persevere in solving problems. The examples below provide students with opportunities to develop expertise in the following Common Core State Standards' mathematical practice:

- *Model with mathematics*

4th Grade



8th Grade



12th Grade



OUTCOME: Students learn about the use of mathematics in other cultures. They recognize the contributions to mathematics from a variety of cultures and the needs that led to those contributions.

EXAMPLE: Students make a timeline covering the history of the U.S. system of measurement with the history of the metric system. They learn how to convert between the two systems and discuss the merits of using the metric system. Students work online with students in a country that uses the metric system and compare their different ways of measuring distance, volume, weight, and temperature.



EXAMPLE: To learn about the euro, students can investigate the following: what denominations of banknotes and coins are available; the euro's value in relation the U.S. dollar; which countries use the euro; when the euro was adopted and why; and how it links the economies of different countries in Europe.



EXAMPLE: Students investigate the geometric patterns in Medieval Islamic tiling. They explore the use of the "girih" in art and architecture and its similarity to modern mathematical tiling patterns known as Penrose tiles. Students also examine the symmetric patterns of the Alhambra, a walled city and fortress in Granada, Spain built under Muslim rule.



EXAMPLE: Students work in groups to investigate the development of mathematics in an ancient culture. Each group focuses on mathematics in a particular culture, such as Babylonian, Indian, Arabic, Chinese, or Mayan mathematics. Students use sources such as the MacTutor History of Mathematics (www-history.mcs.st-and.ac.uk). Each group presents a lesson to the class using the mathematics of the culture studied.



EXAMPLE: Students create a website about the history of trigonometry, focusing on how advancements emerged from practical interests, such as the quest for astronomical measurements, the need to find ways of telling time, and the importance of cartography and navigation tools. Working in groups, each team of students focuses on a different part of the website: One group prepares a report on the development of sine, cosine, and versine in India and how these concepts developed from Indian astronomy. Another group focuses on the further development of trigonometry in the Islamic world and the contributions of Abu Wafa in the 10th century C.E. A third group focuses on work of Al-Biruni the 11th century, including his demonstration of the tangent formula. A fourth group reports on how the work of Jabir Ibn Aflah helped spread trigonometry to Europe in the 13th century.



Social and Cross-Cultural Skills (continued)



4th Grade



8th Grade



12th Grade

OUTCOME: Students apply tools of mathematics, such as statistical analysis, to understanding cross-cultural problems and issues.

EXAMPLE: Students talk with members of the community who depend on local bus service to get to work and find out how well the bus schedule and bus routes support their need to get to work or to meet their children after school. Students then share the community members' feedback with one another, look at local bus schedules and routes, and analyze the situation mathematically. Students prepare a written report to share with the community and other appropriate parties, such as local government officials.



EXAMPLE: Students collect statistics on per capita energy use in India and in the United States to discover that Indians use a lot less energy per capita than Americans. They research lifestyles in India and identify reasons why people use less energy. Students then research population growth rates to discover that India's population is growing much faster than the U.S.. Students create a mathematical formula based on per capita energy use and population growth to compute each country's rate of increase in total energy consumption.



EXAMPLE: Students make a timeline covering the history of different currencies and work out conversions for current values using proportionality and linear functions. They study inflation in terms of exponential growth and use the Cost Performance Index to set up ratios and determine the real value of money. Students read news articles from around the world about currency trading and identify its effects on inflation in various countries.



Productivity and Accountability

Common Core State Standards Alignment:

As the [P21 Common Core Toolkit](#) states, Life and Career skills such as Productivity and Accountability align well with the Common Core State Standards' mathematical practice calling for students to make sense of and persevere in solving problems. The examples below provide students with opportunities to develop expertise in the following Common Core State Standards' mathematical practice:

- *Attend to precision*

4th Grade



OUTCOME: Students set goals, establish priorities and schedules, and meet goals to complete a project.

EXAMPLE: Students work in groups to solve a mathematics problem. Each group is assigned a problem on a different topic and identifies the mathematical knowledge needed to solve it. Groups are then reorganized so that in each new group, each student is responsible for reviewing the mathematics for the problem they worked on in the first group, presenting a solution to their problem to the others in the group. Each student is likewise responsible for helping others in their new group understand their topic.

8th Grade



EXAMPLE: Students work in small groups to research a mathematical topic the class has not yet covered. Students make plans for sharing the work, making sure that each team member understands all aspects of the topic assigned. Each team creates a presentation to teach the rest of the class about their assigned topic. Students focus not only on the content of their presentation, but also on using effective presentation techniques, such as speaking in an appropriate voice, making eye contact, and incorporating visual representations of the mathematical topic. Students use technology in their presentation where appropriate.

12th Grade



EXAMPLE: Students work in small groups to research a mathematical topic the class has not yet covered. Each group explores a different topic, and students are responsible for making sure each team member completely understands the group's topic. Group members work together to teach and present their topic to the rest of the class. Students not only use good presentation techniques and incorporate representations of the mathematical topic, but they also involve the audience by encouraging class members to answer questions, ask questions, share ideas, and solve problems. Students use appropriate technology in their presentations, such as applets, websites, mathematical software, or other presentation tools.



Leadership and Responsibility

Common Core State Standards Alignment:

As the [P21 Common Core Toolkit](#) states, Life and Career skills such as Leadership and Responsibility align well with the Common Core State Standards' mathematical practice calling for students to make sense of and persevere in solving problems. The examples below provide students with opportunities to develop expertise in the following Common Core State Standards' mathematical practices:

- *Construct viable arguments and critique the reasoning of others*
- *Model with mathematics*

4th Grade



OUTCOME: Students use interpersonal and problem-solving skills to leverage strengths of peers and solve mathematical problems important to their community.

EXAMPLE: Students plan a community garden by computing the perimeter and area of the space available and planning what items to plant, based on the local climate, demand for the crop, and how well each item would fit in the available space. Students consider the size and shape of the space devoted to each crop to create a scale drawing of their plan.



8th Grade



EXAMPLE: Students work together to redesign a school playground or public park in their community, and create a scale model of the new design for public view. First, students discuss how the space is currently used and how they think space should be reallocated and redesigned to enhance popular activities or create additional space for activities to benefit the community. They measure each section of the current area, propose design changes, and negotiate with one another on such questions as: whether to enlarge a soccer field; whether to transform a tennis court into a paved space for young children to ride on tricycles; how to improve drainage to create enough space to add a basketball hoop; or how to create an area that would be fun for doing skateboard tricks. Students also research the costs of various proposals and try to minimize costs. Once students reach a consensus on the overall plan and compute necessary dimensions, they divide up the work of creating a scale model of the new design. One team of students creates a two-dimensional scale drawing based on actual measurements of the existing space; another team uses materials such as cardboard, wood, and fabric to create items such as goal posts, playground equipment, also sized to scale. They display the scale model in a central location at the school or in a community building.



12th Grade



EXAMPLE: Students work collaboratively to assess the costs, benefits, and drawbacks of introducing a lottery in the community. They research how the proceeds would be used, how much income the lottery would need to generate to meet its goals, how much to charge for lottery tickets in order to meet income goals, how many winning tickets to offer, and how much to pay out for each winning ticket. Students consider how to make the lottery worthwhile for individual ticket purchasers, calculate the odds of winning, and compute the lottery's value for people who regularly buy tickets. They investigate the reasons why people purchase lottery tickets and debate the ethics of introducing a lottery. Students create informative posters or a website/wiki to educate other students about the long-term expected value of lotteries.



Leadership and Responsibility (continued)

4th Grade



8th Grade



12th Grade



OUTCOME: Students consider the ethical implications of mathematically-based decisions.

EXAMPLE: Students work in small groups in which each team represents a company that derives its profits from the extraction or use of natural resources. For example, one group might act as a large farm in an arid area of the Southwest that uses scarce water resources for irrigation. Another might be a real estate developer in a fragile coastal area, or an oil company that drills beneath the deep sea. Each group plays the role of the company's board of directors and tries to come up with ethical policies for balancing profit and operation costs with social equity and environmental sustainability. Each group presents its policy proposals to the class.



EXAMPLE: Students investigate the “equal proportions” method currently used to determine how U.S. congressional districts are allocated. They examine how the number of congressional districts (and hence, the number of U.S. Representatives) in a state can increase or decrease as districts are “reapportioned” after each decennial census.



Then, students look at the mathematics of how states redraw the boundaries between federal congressional districts in a process that occurs after the number of districts in the state rises or falls. This process is known as “redistricting.”

Next, students look at how their own state's total number of federal congressional districts has been determined, based on U.S. Census data, and changes based on the 2010 Census. Students examine how U.S. congressional district boundaries are drawn in their state. Working in small groups, students identify legal or ethical issues that have arisen. For example, a redistricting plan might have affected the voting representation of a particular minority population.

EXAMPLE: Students examine situations involving systemic risk—in which a company, a government, or other organization has grown “too big to fail” because its failure would cause other companies or organizations to face hardship or collapse, or would cause too many people to panic. Students look at how leaders of large companies such as investment banks have taken advantage of such a situation by taking risks that bring financial gain to the company even if those risks undermine the security of the community, the country, or the world economy. Students propose ways to limit systemic risk.



Credits

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