Name ______________________ Class ____________ Date ________

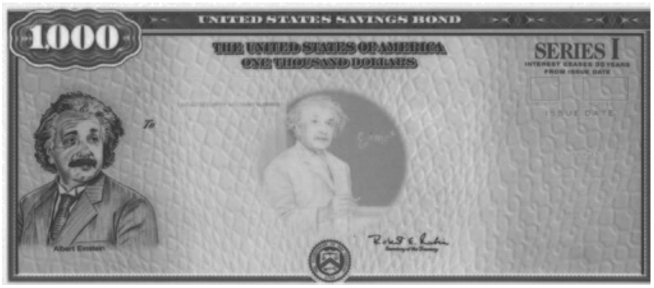
13.4 Compound Interest

Essential Question: How do you model the value of an investment that earns compound interest?

Resource Locker

Explore Comparing Simple and Compound Interest

A *bond* is a type of investment that you buy with cash and for which you receive interest either as the bond matures or when it matures. A *conventional bond* generates an interest payment, sometimes called a *coupon payment*, on a regular basis, typically twice a year. The interest payments end when the bond matures and the amount you paid up-front is returned to you.



A *zero-coupon bond*, on the other hand, requires you to invest less money up-front and pays you its maturity value, which includes all accumulated interest, when the bond matures. Over the life of the bond, the interest that is earned each period itself earns interest until the bond matures.

The basic difference between a conventional bond and a zero-coupon bond is the type of interest earned. A conventional bond pays simple interest, whereas a zero-coupon bond pays **compound interest**, which is interest that earns interest.

Ⓐ For \$1000, you can buy a conventional bond that has a maturity value of \$1000, has a maturity date of 4 years, and pays 5% annual interest. Calculate the interest that the investment earns annually by completing the table. (Bear in mind that interest earned is paid to you and not reinvested.)

Conventional Bond			
Year	**Value of Investment at Beginning of Year**	**Interest Earned for Year (Paid to the Investor)**	**Value of Investment at End of Year**
0			\$1000
1	\$1000	\$1000(0.05) = \$☐	\$☐
2	\$1000	\$1000(0.05) = \$☐	\$☐
3	\$1000	\$1000(0.05) = \$☐	\$☐
4	\$1000	\$1000(0.05) = \$☐	\$☐

(B) For \$822.70, you can buy a zero-coupon bond that has a maturity value of \$1000, has a maturity date of 4 years, and pays 5% annual interest. Calculate the interest that the investment earns annually by completing the table. (Bear in mind that interest earned is reinvested and not paid to you until the bond matures.)

Zero-Coupon Bond			
Year	Value of Investment at Beginning of Year	Interest Earned for Year (Reinvested)	Value of Investment at End of Year
0			\$822.70
1	\$822.70	\$822.70(0.05) = \$ ☐	\$822.70 + \$ ☐ = \$ ☐
2	\$ ☐	\$ ☐ (0.05) = \$ ☐	\$ ☐ + \$ ☐ = \$ ☐
3	\$ ☐	\$ ☐ (0.05) = \$ ☐	\$ ☐ + \$ ☐ = \$ ☐
4	\$ ☐	\$ ☐ (0.05) = \$ ☐	\$ ☐ + \$ ☐ = \$1000

Reflect

1. Describe the difference between how simple interest is calculated and how compound interest is calculated.

2. If $V(t)$ represents the value of an investment at time t, in whole numbers of years starting with $t = 0$ and ending with $t = 4$, write a constant function for the last column in the first table and an exponential function for the last column in the second table.

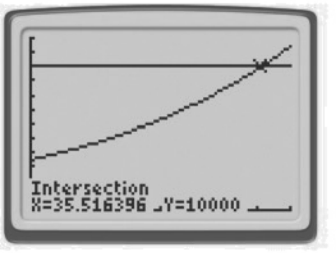
Explain 1 Modeling Interest Compounded Annually

Recall that the general exponential growth function is $f(t) = a(1 + r)^t$. When this function is applied to an investment where interest is compounded annually at a rate r, the function is written as $V(t) = P(1 + r)^t$ where $V(t)$ is the value V of the investment at time t and P is the *principal* (the amount invested).

Ⓐ A person invests $3500 in an account that earns 3% annual interest. Find when the value of the investment reaches $10,000.

Let $P = 3500$ and $r = 0.03$. Then $V(t) = P(1 + r)^t = 3500(1.03)^t$.

Graph $y = 3500(1.03)^x$ on a graphing calculator. Also graph the line $y = 10{,}000$ and find the point where the graphs intersect.



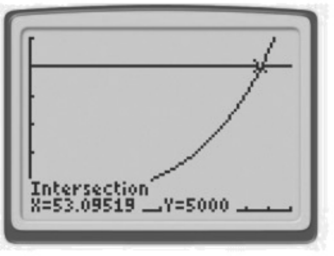
The value of the function is 10,000 at $x \approx 35.5$. So, the investment reaches a value of $10,000 in approximately 35.5 years.

Ⓑ A person invests $200 in an account that earns 6.25% annual interest. Find when the value of the investment reaches $5000.

Let $P =$ ☐ and $r =$ ☐. Then $V(t) = P(1 + r)^t =$ ☐ (☐)^t.

Graph the function on a graphing calculator. Also graph the line

$y =$ ☐ and find the point of intersection.



The value of the function is ________ at $x \approx$ ☐. So, the investment reaches a value of $5000 in approximately ________ years.

Your Turn

3. A person invests $1000 in an account that earns 5.5% annual interest. Find when the value of the investment has doubled.

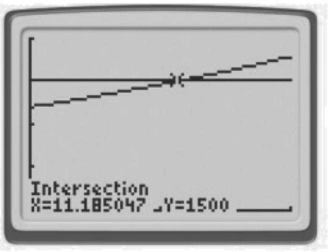
Explain 2 Modeling Interest Compounded More than Once a Year

Interest may be earned more frequently than once a year, such as semiannually (every 6 months), quarterly (every 3 months), monthly, and even daily. If the number of times that interest is compounded in a year is n, then the interest rate per compounding period is $\frac{r}{n}$ (where r is the annual interest rate), and the number of times that interest is compounded in t years is nt. So, the exponential growth function becomes $V(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.

(A) A person invests \$1200 in an account that earns 2% annual interest compounded quarterly. Find when the value of the investment reaches \$1500.

Let $P = 1200$, $r = 0.02$, and $n = 4$. Then $V(t) = P\left(1 + \frac{r}{n}\right)^{nt} = 1200\left(1 + \frac{0.02}{4}\right)^{4t} = 1200(1.005)^{4t}$.

Graph the function $y = 1200(1.005)^{4x}$ on a graphing calculator. Also graph the line $y = 1500$ and find the point where the graphs intersect.



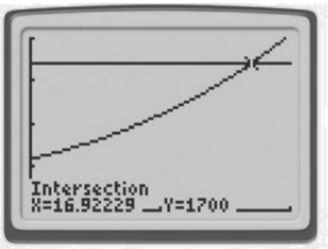
The value of the function is 1500 at $x \approx 11.2$. So, the investment reaches a value of \$1500 in approximately 11.2 years.

(B) A person invests \$600 in an account that earns 6.25% annual interest compounded semiannually. Find when the value of the investment reaches \$1700.

Let $P = \square$, $r = \square$, and $n = \square$.

Then $V(t) = P\left(1 + \frac{r}{n}\right)^{nt} = \square\left(1 + \frac{\square}{\square}\right)^{\square t} = \square\left(\square\right)^{\square t}$.

Graph the function on a graphing calculator. Also graph the line $y = \square$ and find the point where the graphs intersect.



The value of the function is ________ at $x \approx \square$. So, the investment reaches a value of \$1700 in approximately ________ years.

Your Turn

4. A person invests $8000 in an account that earns 6.5% annual interest compounded daily. Find when the value of the investment reaches $20,000.

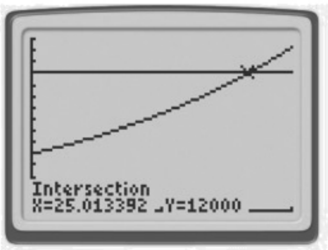
Explain 3 Modeling Interest Compounded Continuously

By letting $m = \frac{n}{r}$, you can rewrite the model $V(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ as $V(t) = P\left(1 + \frac{1}{m}\right)^{mrt}$ because $\frac{r}{n} = \frac{1}{m}$ and $nt = mrt$. Then, rewriting $V(t) = P\left(1 + \frac{1}{m}\right)^{mrt}$ as $V(t) = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$ and letting n increase without bound, which causes m to increase without bound, you see that $\left(1 + \frac{1}{m}\right)^{m}$ approaches e, and the model simply becomes $V(t) = Pe^{rt}$. This model gives the value of an investment with principal P and annual interest rate r when interest is compounded *continuously*.

(A) A person invests $5000 in an account that earns 3.5% annual interest compounded continuously. Find when the value of the investment reaches $12,000.

Let $P = 5000$ and $r = 0.035$. Then $V(t) = 5000e^{0.035t}$.

Graph $y = 5000e^{0.035x}$. on a graphing calculator. Also graph the line $y = 12{,}000$ and find the point where the graphs intersect.

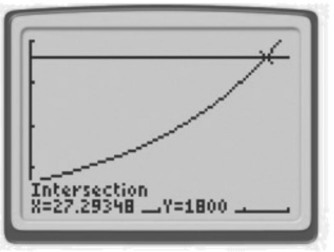


The value of the function is 12,000 at $x \approx 25$. So, the investment reaches a value of $12,000 in approximately 25 years.

(B) The principal amount, \$350, earns 6% annual interest compounded continuously. Find when the value reaches \$1800.

Let $P =$ ☐ and $r =$ ☐. Then $V(t) =$ ☐ $e^{☐ t}$.

Graph the function on a graphing calculator. Also graph the line $y =$ ☐ and find the point where the graphs intersect.



The value of the function is ______ at $x \approx$ ☐. So, the investment reaches a value of \$1800 in approximately ______ years.

Your Turn

5. A person invests \$1550 in an account that earns 4% annual interest compounded continuously. Find when the value of the investment reaches \$2000.

Explain 4 Finding and Comparing Effective Annual Interest Rates

The value-of-an-investment function $V(t) = P\left(1 + \frac{r}{n}\right)^{nt} = P\left[\left(1 + \frac{r}{n}\right)^{n}\right]$^$t$, where interest is compounded n times per year, is an exponential function of the form $f(t) = ab^t$ where $a = P$ and $b = \left(1 + \frac{r}{n}\right)^{n}$. When the base of an exponential function is greater than 1, the function is an exponential growth function where the base is the growth factor and 1 less than the base is the growth rate. So, the growth rate for $V(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ is $b - 1 = \left(1 + \frac{r}{n}\right)^{n} - 1$. This growth rate is called the investment's *effective annual interest rate R*, whereas r is called the investment's *nominal annual interest rate*.

Similarly, for the value-of-an-investment function $V(t) = Pe^{rt} = P\left[e^{r}\right]^{t}$, where interest is compounded continuously, the growth factor is e^r, and the growth rate is $e^r - 1$. So, in this case, the effective annual interest rate is $R = e^r - 1$.

For an account that earns interest compounded more than once a year, the effective annual interest rate is the rate that would produce the same amount of interest if interest were compounded annually instead. The effective rate allows you to compare two accounts that have different nominal rates and different compounding periods.

(A) Arturo plans to make a deposit in one of the accounts shown in the table. To maximize the interest that the account earns, which account should he choose?

	Account X	Account Y
Nominal Annual Interest Rate	2.5%	2.48%
Compounding Period	Quarterly	Monthly

For Account X, interest is compounded quarterly, so $n = 4$. The nominal rate is 2.5%, so $r = 0.025$.

$R_X = \left(1 + \frac{r}{n}\right)^n - 1$ Use the formula for the effective rate.

$= \left(1 + \frac{0.025}{4}\right)^4 - 1$ Substitute.

≈ 0.02524 Simplify.

For Account Y, interest is compounded monthly, so $n = 12$. The nominal rate is 2.48%, so $r = 0.0248$.

$R_Y = \left(1 + \frac{r}{n}\right)^n - 1$ Use the formula for the effective rate.

$= \left(1 + \frac{0.0248}{12}\right)^{12} - 1$ Substitute.

≈ 0.02508 Simplify.

Account X has an effective rate of 2.524%, and Account Y has an effective rate of 2.508%, so Account X has a greater effective rate, and Arturo should choose Account X.

(B) Harriet plans to make a deposit in one of two accounts. Account A has a 3.24% nominal rate with interest compounded continuously, and Account B has a 3.25% nominal rate with interest compounded semiannually. To maximize the interest that the account earns, which account should she choose?

For Account A, interest is compounded continuously. The nominal rate is 3.24%, so $r =$ ☐.

$R_A = e^r - 1$ Use the formula for the effective rate.

$= e^{☐} - 1$ Substitute.

$\approx$ ☐ Simplify.

For Account B, interest is compounded semiannually, so so $n =$ ☐. The nominal rate is 3.25%,

so $r =$ ☐

$$R_B = \left(1 + \frac{r}{n}\right)^n - 1$$ Use the formula for the effective rate.

$$= \left(1 + \frac{\square}{\square}\right)^{\square} - 1$$ Substitute.

$$\approx \square$$ Simplify.

Account A has an effective rate of _______%, and Account B has an effective rate of _______%,

so Account _______ has a greater effective rate, and Harriet should choose Account _______.

Your Turn

6. Jaclyn plans to make a deposit in one of the accounts shown in the table. To maximize the interest that the account earns, which account should she choose?

	Account X	Account Y
Nominal Annual Interest Rate	4.24%	4.18%
Compounding Period	Annually	Daily

Elaborate

7. Explain the difference between an investment's nominal annual interest rate and its effective annual interest rate.

8. **Essential Question Check-In** List the three functions used to model an investment that earns compound interest at an annual rate *r*. Identify when each function is used.

Evaluate: Homework and Practice

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1. A person invests $2560 in an account that earns 5.2% annual interest. Find when the value of the investment reaches $6000.

2. A person invests $1800 in an account that earns 2.46% annual interest. Find when the value of the investment reaches $3500.

3. Emmanuel invests $3600 and Kelsey invests $2400. Both investments earn 3.8% annual interest. How much longer will it take Kelsey's investment to reach $10,000 than Emmanuel's investment?

4. Jocelyn invests $1200 in an account that earns 2.4% annual interest. Marcus invests $400 in an account that earns 5.2% annual interest. Find when the value of Marcus's investment equals the value of Jocelyn's investment and find the common value of the investments at that time.

5. A person invests $350 in an account that earns 3.65% annual interest compounded semiannually. Find when the value of the investment reaches $5675.

6. Molly invests \$8700 into her son's college fund, which earns 2% annual interest compounded daily. Find when the value of the fund reaches \$12,000.

7. A person invests \$200 in an account that earns 1.98% annual interest compounded quarterly. Find when the value of the investment reaches \$500.

8. Hector invests \$800 in an account that earns 6.98% annual interest compounded semiannually. Rebecca invests \$1000 in an account that earns 5.43% annual interest compounded monthly. Find when the value of Hector's investment equals the value of Rebecca's investment and find the common value of the investments at that time.

9. A person invests \$6750 in an account that earns 6.23% annual interest compounded continuously. Find when the value of the investment reaches \$15,000.

10. A person invests \$465 in an account that earns 3.1% annual interest compounded continuously. Find when the value of the investment reaches \$2400.

11. Lucy invests \$800 in an account that earns 6.12% annual interest compounded continuously. Juan invests \$1600 in an account that earns 3.9% annual interest compounded continuously. Find when the value of Lucy's investment equals the value of Juan's investment and find the common value of the investments at that time.

12. Paula plans to make a deposit in one of the accounts shown in the table. To maximize the interest that the account earns, which account should she choose?

	Account X	Account Y
Nominal Annual Interest rate	2.83%	2.86%
Compounding Period	Continuously	Annually

13. Tanika plans to make a deposit to one of two accounts. Account A has a 3.78% nominal rate with interest compounded daily, and Account B has a 3.8% nominal rate with interest compounded monthly. To maximize the interest that the account earns, which account should she choose?

14. Kylie plans to deposit $650 in one of the accounts shown in the table. She chooses the account with the greater effective rate. How much money will she have in her account after 10 years?

	Account X	Account Y
Nominal Annual Interest Rate	4.13%	4.12%
Compounding Period	Semiannually	Quarterly

15. A person invests $2860 for 15 years in an account that earns 4.6% annual interest. Match each description of a difference in interest earned on the left with the actual difference listed on the right.

A. Difference between compounding interest semiannually and annually	_______ $14.89
B. Difference between compounding interest quarterly and semiannually	_______ $21.98
C. Difference between compounding interest monthly and quarterly	_______ $0.25
D. Difference between compounding interest daily and monthly	_______ $42.75
E. Difference between compounding interest continuously and daily	_______ $7.27

H.O.T. Focus on Higher Order Thinking

16. Multi-Step Ingrid and Harry are saving to buy a house. Ingrid invests $5000 in an account that earns 3.6% interest compounded quarterly. Harry invests $7500 in an account that earns 2.8% interest compounded semiannually.

a. Find a model for each investment.

b. Use a graphing calculator to find when the combined value of their investments reaches $15,000.

17. Explain the Error A student is asked to find when the value of an investment of $5200 in an account that earns 4.2% annual interest compounded quarterly reaches $16,500. The student uses the model $V(t) = 5200(1.014)^{3t}$ and finds that the investment reaches a value of $16,500 after approximately 27.7 years. Find and correct the student's error.

18. Communicate Mathematical Ideas For a certain price, you can buy a zero-coupon bond that has a maturity value of $1000, has a maturity date of 4 years, and pays 5% annual interest. How can the present value (the amount you pay for the bond) be determined from the future value (the amount you get when the bond matures)?

Lesson Performance Task

The grandparents of a newborn child decide to establish a college fund for her. They invest \$10,000 into a fund that pays 4.5% interest compounded continuously.

a. Write a model for the value of the investment over time and find the value of the investment when the child enters college at 18 years old.

b. In 2013, the average annual public in-state college tuition was \$8893, which was 2.9% above the 2012 cost. Use these figures to write a model to project the amount of money needed to pay for one year's college tuition 18 years into the future. What amount must be invested in the child's college fund to generate enough money in 18 years to pay for the first year's college tuition?