Recommendation 5. Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas.

A major problem for students who struggle with mathematics is weak understanding of the relationships between the abstract symbols of mathematics and the various visual representations.[93] Student understanding of these relationships can be strengthened through the use of visual representations of mathematical concepts such as solving equations, fraction equivalence, and the commutative property of addition and multiplication (see the glossary). Such representations may include number lines, graphs, simple drawings of concrete objects such as blocks or cups, or simplified drawings such as ovals to represent birds.

In the view of the panel, the ability to express mathematical ideas using visual representations and to convert visual representations into symbols is critical for success in mathematics. A major goal of interventions should be to systematically teach students how to develop visual representations and how to transition these representations to standard symbolic representations used in problem solving. Occasional and unsystematic exposure (the norm in many classrooms) is insufficient and does not facilitate understanding of the relationship between the abstract symbols of mathematics and various visual representations.

93. Hecht, Vagi, and Torgesen (2007).

Level of evidence: Moderate

The panel judged the level of evidence supporting this recommendation to be *moderate*. This recommendation is based on 13 randomized controlled trials that met WWC standards or met standards with reservations.[94] These studies provide support for the systematic use of visual representations or manipulatives to improve achievement in general mathematics,[95] prealgebra concepts,[96] word problems,[97] and operations.[98] But these representations were part of a complex multicomponent intervention in each of the studies. So, it is difficult to judge the impact of the representation component alone, and the panel believes that a *moderate* designation is appropriate for the level of evidence for this recommendation.

Brief summary of evidence to support the recommendation

Research shows that the systematic use of visual representations and manipulatives may lead to statistically significant or substantively important positive gains in math

94. Artus and Dyrek (1989); Butler et al. (2003); Darch, Carnine, and Gertsen (1984); Fuchs et al. (2005); Fuchs, Seethaler et al. (2008); Fuchs, Powell et al. (2008); Fuchs, Fuchs, Craddock et al. (2008); Jitendra et al. (1998); Walker and Poteet (1989); Wilson and Sindelar (1991); Witzel (2005); Witzel, Mercer, and Miller (2003); Woodward (2006).

95. Artus and Dyrek (1989); Fuchs et al. (2005).

96. Witzel, Mercer, and Miller (2003).

97. Darch, Carnine, and Gersten (1984); Fuchs et al. (2005); Fuchs, Seethaler et al. (2008); Fuchs, Fuchs, Craddock et al. (2008); Jitendra et al. (1998); Wilson and Sindelar (1991).

98. Woodward (2006).

achievement.[99] Four studies used visual representations to help pave the way for students to understand the abstract version of the representation.[100] For example, one of the studies taught students to use visual representations such as number lines to understand mathematics facts.[101] The four studies demonstrated gains in mathematics facts and operations[102] and word problem proficiencies,[103] and may provide evidence that using visual representations in interventions is an effective technique.

Three of the studies used manipulatives in the early stages of instruction to reinforce understanding of basic concepts and operations.[104] One used concrete models such as groups of boxes to teach rules for multiplication problems.[105] The three studies largely showed significant and positive effects and provide evidence that using manipulatives may be helpful in the initial stages of an intervention to improve proficiency in word problem solving.[106]

In six of the studies, both concrete and visual representations were used, and overall these studies show that using some combination of manipulatives and visual representations may promote mathematical understanding.[107] In two of the six, instruction did not include fading of the manipulatives and visual representations to promote understanding of math at a more abstract level.[108] One of these interventions positively affected general math achievement,[109] but the other had no effect on outcome measures tested.[110] In the other four studies, manipulatives and visual representations were presented to the students sequentially to promote understanding at a more abstract level.[111] One intervention that used this method for teaching fractions did not show much promise,[112] but the other three did result in positive gains.[113] One of them taught 1st graders basic math concepts and operations,[114] and the other two taught prealgebra concepts to low-achieving students.[115]

How to carry out this recommendation

1. Use visual representations such as number lines, arrays, and strip diagrams.

In the panel's view, visual representations such as number lines, number paths, strip diagrams, drawings, and other forms of pictorial representations help scaffold learning and pave the way for understanding the abstract version of the representation. We recommend that interventionists use such abstract visual representations extensively and consistently. We also recommend that interventionists explicitly link visual representations with the standard symbolic representations used in mathematics.

99. Following WWC guidelines, an effect size greater than 0.25 is considered substantively important.

100. Jitendra et al. (1998); Walker and Poteet (1989); Wilson and Sindelar (1991); Woodward (2006).

101. Woodward (2006).

102. Woodward (2006).

103. Jitendra et al. (1998); Walker and Poteet (1989); Wilson and Sindelar (1991).

104. Darch et al. (1984); Fuchs, Seethaler et al. (2008); Fuchs, Fuchs, Craddock et al. (2008).

105. Darch et al. (1984).

106. Darch at al. (1984); Fuchs, Sethaler et al. (2008); Fuchs, Fuchs, Craddock et al. (2008).

107. Artus and Dyrek (1989); Butler et al. (2003); Fuchs, Powell et al. (2008); Fuchs et al. (2005); Witzel (2005); Witzel, Mercer, and Miller (2003).

108. Artus and Dyrek (1989); Fuchs, Powell et al. (2008).

109. Artus and Dyrek (1989).

110. Fuchs, Powell et al. (2008).

111. Fuchs et al. (2005); Butler et al. (2003); Witzel et al. (2003); Witzel (2005).

112. Butler et al. (2003).

113. Fuchs et al. (2005); Witzel, Mercer, and Miller (2003); Witzel (2005).

114. Fuchs et al. (2005).

115. Witzel, Mercer, and Miller (2003); Witzel (2005).

In early grades, number lines, number paths, and other pictorial representations are often used to teach students foundational concepts and procedural operations of addition and subtraction. Although number lines or number paths may not be a suitable initial representation in some situations (as when working with multiplication and division), they can help conceptually and procedurally with other types of problems. Conceptually, number lines and number paths show magnitude and allow for explicit instruction on magnitude comparisons. Procedurally, they help teach principles of addition and subtraction operations such as "counting down," "counting up," and "counting down from."

The figure in example 4 shows how a number line may be used to assist with counting strategies. The top arrows show how a child learns to count on. He adds 2 + 5 = ___. To start, he places his finger on 2. Then, he jumps five times to the right and lands on 7. The arrows under the number line show how a child subtracts using a counting down strategy. For 10 – 3 = ____, she starts with her finger on the 10. Then, she jumps three times to the left on the number line, where she finishes on 7.

The goal of using a number line should be for students to create a mental number line and establish rules for movement along the line according to the more or less marking arrows placed along the line. Such rules and procedures should be directly tied to the explicit instruction that guided the students through the use of the visual representation.[116]

Pictorial representations of objects such as birds and cups are also often used to teach basic addition and subtraction, and simple drawings can help students understand place value and multidigit addition and subtraction. Example 5 (p. 34) shows how a student can draw a picture to solve a multidigit addition problem. In the figure, circles represent one unit and lines represent units of 10.

In upper grades, diagrams and pictorial representations used to teach fractions also help students make sense of the basic structure underlying word problems. Strip diagrams (also called model diagrams and bar diagrams) are one type of diagram that can be used. Strip diagrams are drawings of narrow rectangles that show relationships among quantities. Students can use strip diagrams to help them reason about and solve a wide variety of word problems about related quantities. In example 6 (p. 34), the full rectangle (consisting of all three equal parts joined together) represents Shauntay's money before she bought the book. Since she spent ⅔ of her money on the book, two of the three equal parts represent the $26 she spent on the book. Students can then reason that if two parts stand for $26, then each part stands for $13, so three parts stand for $39. So, Shauntay had $39 before she bought the book.

2. If visuals are not sufficient for developing accurate abstract thought and answers, use concrete manipulatives first. Although this can also be done with students in upper elementary and middle school grades, use of manipulatives with older students should be expeditious because the goal is to move toward understanding of—and facility with—visual representations, and finally, to the abstract.

Manipulatives are usually used in lower grades in the initial stages of learning as teachers introduce basic concepts with whole numbers. This exposure to concrete objects is often fleeting and transitory. The use of manipulatives in upper elementary school grades is virtually nonexistent.[117]

116. Manalo, Bunnell, and Stillman (2000). Note that this study was not eligible for review because it was conducted outside the United States.

117. Howard, Perry, and Lindsay (1996); Howard, Perry, and Conroy (1995).

The panel suggests that the interventionist use concrete objects in two ways.

First, in lower elementary grades, use concrete objects more extensively in the initial stages of learning to reinforce the understanding of basic concepts and operations.[118]

Concrete models are routinely used to teach basic foundational concepts such as place value.[119] They are also useful in teaching other aspects of mathematics such as multiplication facts. When a multiplication fact is memorized by question and answer alone, a student may believe that numbers are to be memorized rather than understood. For example, 4×6 equals 24. When shown using manipulatives (as in example 7, p. 35), 4×6 means 4 groups of 6, which total as 24 objects.

Second, in the upper grades, use concrete objects when visual representations do not seem sufficient in helping students understand mathematics at the more abstract level.

Use manipulatives expeditiously, and focus on fading them away systematically to reach the abstract level.[120] In other words, explicitly teach students the concepts and operations when students are at the concrete level and consistently repeat the instructional procedures at the visual and abstract levels. Using consistent language across representational systems (manipulatives, visual representations, and abstract symbols) has been an important component in several research studies.[121] Example 8 (p. 35) shows a set of matched concrete, visual, and abstract representations of a concept involving solving single-variable equations.

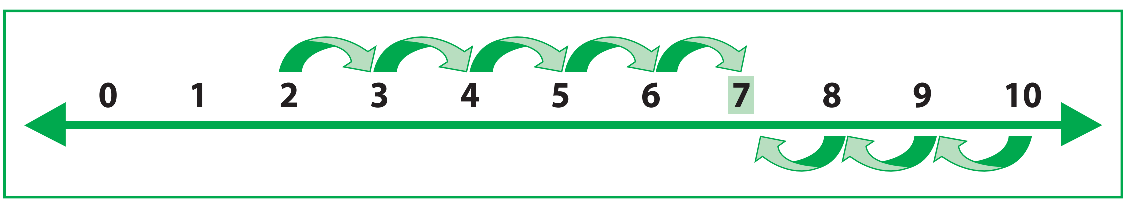
118. Darch at al. (1984); Fuchs, Seethaler et al. (2008) Fuchs, Fuchs, Craddock et al. (2008).

119. Fuchs et al. (2005); Fuchs, Seethaler et al. (2008); Fuchs, Powell et al. (2008).

120. Fuchs et al. (2005); Witzel (2005); Witzel, Mercer, and Miller (2003).

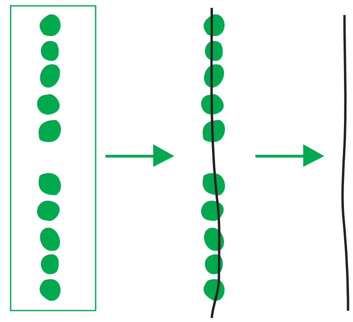
121. Fuchs et al. (2005); Butler et al. (2003); Witzel (2005); Witzel, Mercer, and Miller (2003).

Example 4. Representation of the counting on strategy using a number line

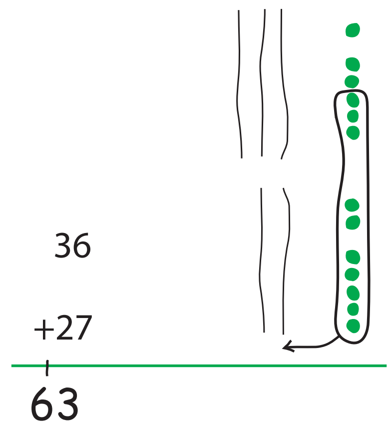


Example 5. Using visual representations for multidigit addition

A group of ten can be drawn with a long line to indicate that ten ones are joined to form one ten:

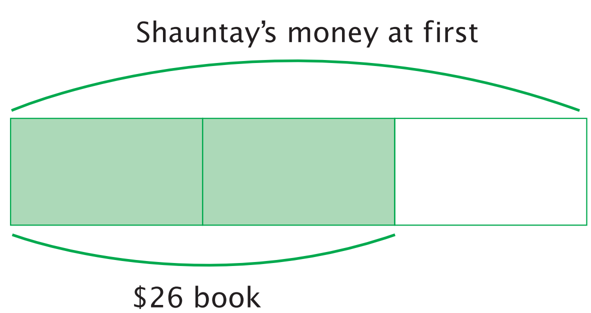


Simple drawings help make sense of two-digit addition with regrouping:



Example 6. Strip diagrams can help students make sense of fractions

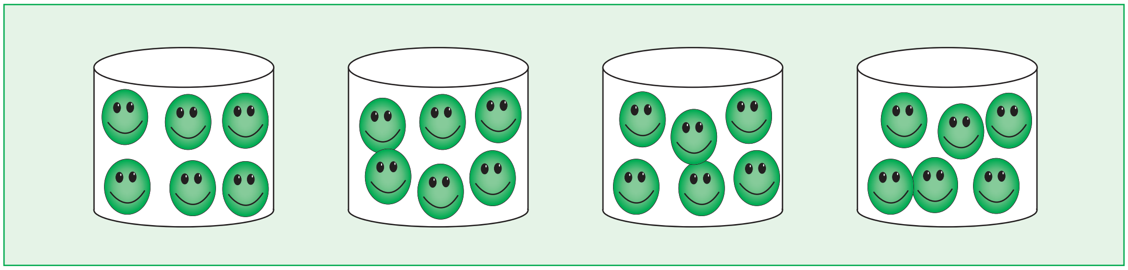
Shauntay spent $^2/_3$ of the money she had on a book that cost \$26. How much money did Shauntay have before she bought the book?



2 parts → \$26
1 part → \$26 ÷ 2 = \$13
3 parts → 3 × \$13 = \$39

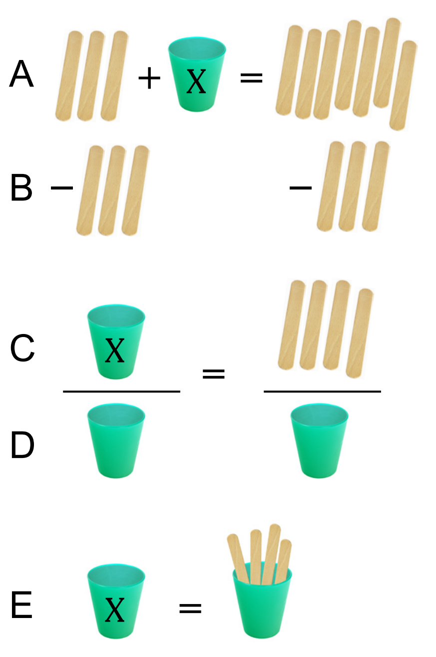
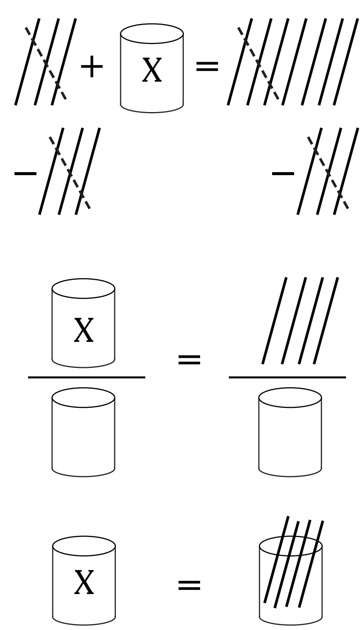
Shauntay's had \$39

Example 7. Manipulatives can help students understand that four multiplied by six means four groups of six, which means 24 total objects



Example 8. A set of matched concrete, visual, and abstract representations to teach solving single-variable equations

3 + X = 7

Solving the Equation with Concrete Manipulatives (Cups and Sticks)	Solving the Equation with Visual Representations of Cups and Sticks	Solving the Equation with Abstract Symbols
		$3 + 1X = 7$ $-3 \quad -3$ $\frac{1X}{1} = \frac{4}{1}$ $X = 4$

Concrete Steps

A. 3 sticks plus one group of X equals 7 sticks
B. Subtract 3 sticks from each side of the equation
C. The equation now reads as one group of X equals 4 sticks
D. Divide each side of the equation by one group
E. One group of X is equal to four sticks (i.e., 1X/group = 4 sticks/group; 1X = 4 sticks)

Potential roadblocks and solutions

Roadblock 5.1. *In the opinion of the panel, many intervention materials provide very few examples of the use of visual representations.*

Suggested Approach. Because many curricular materials do not include sufficient examples of visual representations, the interventionist may need the help of the mathematics coach or other teachers in developing the visuals. District staff can also arrange for the development of these materials for use throughout the district.

Roadblock 5.2. *Some teachers or interventionists believe that instruction in concrete manipulatives requires too much time.*

Suggested Approach. Expeditious use of manipulatives cannot be overemphasized. Since tiered interventions often rely on foundational concepts and procedures, the use of instruction at the concrete level allows for reinforcing and making explicit the foundational concepts and operations. Note that overemphasis on manipulatives can be counterproductive, because students manipulating only concrete objects may not be learning to do math at an abstract level.[122] The interventionist should use manipulatives in the initial stages strategically and then scaffold instruction to the abstract level. So, although it takes time to use manipulatives, this is not a major concern since concrete instruction will happen only rarely and expeditiously.

Roadblock 5.3. *Some interventionists may not fully understand the mathematical ideas that underlie some of the representations. This is likely to be particularly true for topics involving negative numbers, proportional reasoning, and interpretations of fractions.*

Suggested Approach. If interventionists do not fully understand the mathematical ideas behind the material, they are unlikely to be able to teach it to struggling students.[123] It is perfectly reasonable for districts to work with a local university faculty member, high school mathematics instructor, or mathematics specialist to provide relevant mathematics instruction to interventionists so that they feel comfortable with the concepts. This can be coupled with professional development that addresses ways to explain these concepts in terms their students will understand.

122. Witzel, Mercer, and Miller (2003).

123. Hill, Rowan, and Ball (2005); Stigler and Hiebert (1999).

Recommendation 6. Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.

Quick retrieval of basic arithmetic facts is critical for success in mathematics.[124] Yet research has found that many students with difficulties in mathematics are not fluent in such facts.[125] Weak ability to retrieve arithmetic facts is likely to impede understanding of concepts students encounter with rational numbers since teachers and texts often assume automatic retrieval of facts such as $3 \times 9 = \_\_\_\_$ and $11 - 7 = \_\_\_\_$ as they explain concepts such as equivalence and the commutative property.[126] For that reason, we recommend that about 10 minutes be devoted to building this proficiency during each intervention session. Acknowledging that time may be short, we recommend a minimum of 5 minutes a session.

Level of evidence: Moderate

The panel judged the level of evidence supporting this recommendation to be *moderate*. This recommendation is based on seven randomized controlled trials that met WWC standards or met standards with reservations and that included fact fluency instruction in the intervention.[127] These studies reveal a series of small but positive effects on measures of fact fluency[128] and procedural knowledge for diverse student populations in the elementary grades.[129] In some cases, fact fluency instruction was one of several components in the intervention, and it is difficult to judge the impact of the fact fluency component alone.[130] However, because numerous research teams independently produced similar findings, we consider this practice worthy of serious consideration. Although the research is limited to the elementary school grades, in the panel's view, building fact fluency is also important for middle school students when used appropriately.

Brief summary of evidence to support the recommendation

The evidence demonstrates small positive effects on fact fluency and operations for the elementary grades and thus provides support for including fact fluency activities as either stand-alone interventions or components of larger tier 2 interventions.[131] These positive effects did not, however, consistently reach statistical significance, and the findings cannot be extrapolated to areas of mathematics outside of fact fluency and operations.

124. National Mathematics Advisory Panel (2008).

125. Geary (2004); Jordan, Hanich, and Kaplan (2003); Goldman, Pellegrino, and Mertz (1988).

126. Gersten and Chard (1999); Woodward (2006); Jitendra et al. (1996).

127. Beirne-Smith (1991); Fuchs, Seethaler et al. (2008); Fuchs, Fuchs, Hamlett et al. (2006); Fuchs et al. (2005); Fuchs, Powell et al. (2008); Tournaki (2003); Woodward (2006).

128. Fuchs, Seethaler et al. (2008); Fuchs, Fuchs, Hamlet et al. (2006); Fuchs et al. (2005); Fuchs, Powell et al. (2008); Tournaki (2003); Woodward (2006).

129. Beirne-Smith (1991); Fuchs, Seethaler et al. (2008); Fuchs et al. (2005); Tournaki (2003); Woodward (2006).

130. Fuchs, Seethaler et al. (2008); Fuchs et al. (2005).

131. Beirne-Smith (1991); Fuchs et al. (2005); Fuchs, Fuchs, Hamlet et al. (2006); Fuchs, Seethaler et al. (2008); Fuchs, Powell et al. (2008); Tournaki (2003); Woodward (2006).

Two studies examined the effects of being taught mathematics facts relative to the effects of being taught spelling or word identification using similar methods.[132] In both studies, the mathematics facts group demonstrated positive gains in fact fluency relative to the comparison group, but the effects were significant in only one of the studies.[133]

Another two interventions included a facts fluency component in combination with a larger tier 2 intervention.[134] For example, in the Fuchs et al. (2005) study, the final 10 minutes of a 40 minute intervention session were dedicated to practice with addition and subtraction facts. In both studies, tier 2 interventions were compared against typical tier 1 classroom instruction. In each study, the effects on mathematics facts were small and not significant, though the effects were generally positive in favor of groups that received the intervention. Significant positive effects were detected in both studies in the domain of operations, and the fact fluency component may have been a factor in improving students' operational abilities.

Many of the studies in the evidence base included one or more of a variety of components such as teaching the relationships among facts,[135] making use of a variety of materials such as flash cards and computer-assisted instruction,[136] and teaching math facts for a minimum of 10 minutes per session.[137] Since these components were typically not independent variables in the studies, it is difficult to attribute any positive effects to the component itself. There is evidence, however, that strategy-based instruction for fact fluency (such as teaching the counting-on procedure) is superior to rote memorization.[138]

How to carry out this recommendation

1. Provide about 10 minutes per session of instruction to build quick retrieval of basic arithmetic facts. Consider using technology, flash cards, and other materials for extensive practice to facilitate automatic retrieval.

The panel recommends providing about 10 minutes each session for practice to help students become automatic in retrieving basic arithmetic facts, beginning in grade 2. The goal is quick retrieval of facts using the digits 0 to 9 without any access to pencil and paper or manipulatives.

Presenting facts in number families (such as $7 \times 8 = 56$, $8 \times 7 = 56$, $56/7 = 8$, and $56/8 = 7$) shows promise for improving student fluency.[139] In the panel's view, one advantage of this approach is that students simultaneously learn about the nature of inverse operations.

In the opinion of the panel, cumulative review is critical if students are to maintain fluency and proficiency with mathematics facts. An efficient way to achieve this is to integrate previously learned facts into practice activities. To reduce frustration and provide enough extended practice so that retrieval becomes automatic (even for

132. Fuchs, Fuchs, Hamlett et al. (2006); Fuchs, Powell et al. (2008).

133. In Fuchs, Fuchs, Hamlett et al. (2006), the effects on addition fluency were statistically significant and positive while there was no effect on subtraction fluency.

134. Fuchs, Seethaler et al. (2008); Fuchs et al. (2005).

135. Beirne-Smith (1991); Fuchs et al. (2005); Fuchs, Fuchs, Hamlett et al. (2006); Fuchs, Seethaler et al. (2008); Woodward (2006).

136. Beirne-Smith (1991); Fuchs et al. (2005); Fuchs, Fuchs, Hamlett et al. (2006); Fuchs, Seethaler et al. (2008); Fuchs, Powell et al. (2008).

137. Beirne-Smith (1991); Fuchs et al. (2005); Fuchs, Fuchs, Hamlett et al. (2006); Fuchs, Seethaler et al. (2008); Fuchs, Powell et al. (2008); Tournaki (2003); Woodward (2006).

138. Beirne-Smith (1991); Tournaki (2003); Woodward (2006).

139. Fuchs et al. (2005); Fuchs, Fuchs, Hamlett et al. (2006); Fuchs, Seethaler et al. (2008).

those who tend to have limited capacity to remember and retrieve abstract material), interventionists can individualize practice sets so students learn one or two new facts, practice several recently acquired facts, and review previously learned facts.[140] If students are proficient in grade-level mathematics facts, then the panel acknowledges that students might not need to practice each session, although periodic cumulative review is encouraged.

2. For students in kindergarten through grade 2, explicitly teach strategies for efficient counting to improve the retrieval of mathematics facts.

It is important to provide students in kindergarten through grade 2 with strategies for efficiently solving mathematics facts as a step toward automatic, fluent retrieval. The counting-up strategy has been used to increase students' fluency in addition facts.[141] This is a simple, effective strategy that the majority of students teach themselves, sometimes as early as age 4.[142] But students with difficulties in mathematics tend not to develop this strategy on their own, even by grade 2.[143] There is evidence that systematic and explicit instruction in this strategy is effective.[144]

Students can be explicitly taught to find the smaller number in the mathematics fact, put up the corresponding number of fingers, and count up that number of fingers from the larger number. For example, to solve 3 + 5 = ____, the teacher identifies the smaller number (3) and puts up three fingers. The teacher simultaneously says and points to the larger number before counting three fingers, 6, 7, 8.

Note that learning the counting-up strategy not only improves students' fact fluency[145] but also immerses students in the commutative property of addition. For example, students learn that when the larger number is presented second (3 + 5 = ____), they can rearrange the order and start counting up from 5. In the view of the panel, this linkage is an important part of intervention. After this type of instruction, follow-up practice with flash cards might help students make the new learning automatic.

3. Teach students in grades 2 through 8 how to use their knowledge of properties, such as commutative, associative, and distributive law, to derive facts in their heads.

Some researchers have argued that rather than solely relying on rote memorization and drill and practice, students should use properties of arithmetic to solve complex facts involving multiplication and division.[146] These researchers believe that by teaching the use of composition and decomposition, and applying the distributive property to situations involving multiplication, students can increasingly learn how to quickly (if not automatically) retrieve facts. For example, to understand and quickly produce the seemingly difficult multiplication fact $13 \times 7 =$ ____, students are reminded that $13 = 10 + 3$, something they should have been taught consistently during their elementary career. Then, since $13 \times 7 = (10 + 3) \times 7 = 10 \times 7 + 3 \times 7$, the fact is parsed into easier, known problems $10 \times 7 =$ ____ and $3 \times 7 =$ ____ by applying of the distributive property. Students can then rely on the two simpler multiplication facts (which they had already acquired) to quickly produce an answer mentally.

The panel recommends serious consideration of this approach as an option for students who struggle with acquisition of

140. Hasselbring, Bransford, and Goin (1988). Note that there was not sufficient information to do a WWC review.

141. Beirne-Smith (1991); Tournaki (2003).

142. Siegler and Jenkins (1989).

143. Tournaki (2003).

144. Tournaki (2003).

145. Tournaki (2003).

146. Robinson, Menchetti, and Torgesen (2002); Woodward (2006).

facts in grades 2 through 8. When choosing an intervention curriculum, consider one that teaches this approach to students in this age range. Note, however, that the panel believes students should also spend time after instruction with extensive practice on quick retrieval of facts through the use of materials such as flash cards or technology.

Roadblocks and solutions

Roadblock 6.1. *Students may find fluency practice tedious and boring.*

Suggested Approach. Games that provide students with the opportunity to practice new facts and review previously learned facts by encouraging them to beat their previous high score can help the practice be less tedious.[147] Players may be motivated when their scores rise and the challenge increases. Further recommendations for motivating students are in recommendation 8.

Roadblock 6.2. *Curricula may not include enough fact practice or may not have materials that lend themselves to teaching strategies.*

Suggested Approach. Some contemporary curricula deemphasize fact practice, so this is a real concern. In this case, we recommend using a supplemental program, either flash card or technology based.

147. Fuchs, Seethaler et al. (2008).