20.1 Conditional Probability

Essential Question: How do you calculate a conditional probability?

Explore 1 Finding Conditional Probabilities from a Two-Way Frequency Table

The probability that event A occurs given that event B has already occurred is called the **conditional probability** of A given B and is written $P(A|B)$.

One hundred migraine headache sufferers participated in a study of a new medicine. Some were given the new medicine, and others were not. After one week, participants were asked if they had experienced a headache during the week. The two-way frequency table shows the results.

	Took medicine	No medicine	Total
Headache	11	13	24
No headache	54	22	76
Total	65	35	100

Let event A be the event that a participant did not get a headache. Let event B be the event that a participant took the medicine.

(A) To the nearest percent, what is the probability that a participant who took the medicine did not get a headache?

65 participants took the medicine.

Of these, **54** did not get a headache.

So, $P(A|B) = \dfrac{54}{65} \approx$ **83** %.

*pay close attention to how this is written.
-first thing written determines the total*

(B) To the nearest percent, what is the probability that a participant who did not get a headache took the medicine?

76 participants did not get a headache.

Of these, **54** took the medicine.

So, $P(B|A) = \dfrac{54}{76} \approx$ **71** %.

(C) Let $n(A)$ be the number of participants who did not get a headache, $n(B)$ be the number of participants who took the medicine, and $n(A \cap B)$ be the number of participants who took the medicine and did not get a headache.

$n(A) =$ **76** $n(B) =$ **65** $n(A \cap B) =$ **54**

Express $P(A|B)$ and $P(B|A)$ in terms of $n(A)$, $n(B)$, and $n(A \cap B)$.

$P(A|B) = \dfrac{n(A \cap B)}{n(B)}$ $P(B|A) = \dfrac{n(A \cap B)}{n(A)}$

- ∩ means intersection what they have in common.

- ∪ means union everything together

Reflect

1. For the question "What is the probability that a participant who did not get a headache took the medicine?", what event is assumed to have already occurred?

2. In general, does it appear that $P(A|B) = P(B|A)$? Why or why not?

   P(A/B) and P(B/A) are not equal

⊘ Explore 2 Finding Conditional Probabilities from a Two-Way Relative Frequency Table

You can develop a formula for $P(A|B)$ that uses relative frequencies (which are probabilities) rather than frequencies (which are counts).

	Took medicine	No medicine	Total
Headache	11	13	24
No headache	54	22	76
Total	65	35	100

※ when converting to decimals you can do correctly the math way and divide by 100 or you can do your way and move decimal two places to the left ※

Ⓐ To obtain relative frequencies, divide every number in the table by 100, the total number of participants in the study.

	Took medicine	No medicine	Total
Headache	.11	.13	.24
No headache	.54	.22	.76
Total	.65	.35	1

Ⓑ Recall that event A is the event that a participant did not get a headache and that event B is the event that a participant took the medicine. Use the relative frequency table from Step A to find $P(A)$, $P(B)$, and $P(A \cap B)$.

$P(A) = 0.76$, $P(B) = 0.65$ $P(A \cap B) = .54$

Ⓒ In the first Explore, you found the conditional probabilities $P(A|B) \approx 83\%$ and $P(B|A) \approx 71\%$ by using the frequencies in the two-way frequency table. Use the relative frequencies from the table in Step A to find the equivalent conditional probabilities.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.54}{.65} \approx 83\% \qquad P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{.54}{.76} \approx 71\%$$

(D) Generalize the results by using $n(S)$ as the number of elements in the sample space (in this case, the number of participants in the study). For instance, you can write $P(A) = \frac{n(A)}{n(S)}$. Write each of the following probabilities in a similar way.

$$P(B) = \frac{n(B)}{n(S)}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)}$$

$$P(A|B) = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = \frac{P(A \cap B)}{P(B)}$$

- you divide the questions by n(s) to get the probability.

Reflect

3. Why are the two forms of $P(A \cap B)$, $\frac{n(A \cap B)}{n(B)}$ and $\frac{P(A \cap B)}{P(B)}$, equivalent?

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4. What is a formula for $P(B|A)$ that involves probabilities rather than counts? How do you obtain this formula from the fact that $P(B|A) = \frac{n(A \cap B)}{n(A)}$?

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🔵 Explain 1   Using the Conditional Probability Formula

In the previous Explore, you discovered the following formula for conditional probability.

Conditional Probability

The conditional probability of A given B (that is, the probability that event A occurs given that event B occurs) is as follows:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

} make sure you know this

Example 1   Find the specified probability.

(A) For a standard deck of playing cards, find the probability that a red card randomly drawn from the deck is a jack.

4 suits - 13 cards each
-each suits equipped with 4 face cards - (A) K, Q, J and number cards 2-10
→ not always considered a face card b/c has two values.

Step 1 Find $P(R)$, the probability that a red card is drawn from the deck.

There are 26 red cards in the deck of 52 cards, so $P(R) = \frac{26}{52}$.

Step 2 Find $P(J \cap R)$, the probability that a red jack is drawn from the deck.

There are 2 red jacks in the deck, so $P(J \cap R) = \frac{2}{52}$.

Step 3 Substitute the probabilities from Steps 1 and 2 into the formula for conditional probability.

$$P(J|R) = \frac{P(J \cap R)}{P(R)} = \frac{\frac{2}{52}}{\frac{26}{52}}$$

(handwritten) – divide the numerator by the reciprocal.

Step 4 Simplify the result.

$$P(J|R) = \frac{\frac{2}{52} \cdot 52}{\frac{26}{52} \cdot 52} = \frac{2}{26} = \frac{1}{13}$$

(B) For a standard deck of playing cards, find the probability that a jack randomly drawn from the deck is a red card.

Step 1 Find $P(J)$, the probability that a jack is drawn from the deck.

There are **4** jacks in the deck of 52 cards, so $P(J) = \dfrac{4}{52}$.

Step 2 Find $P(J \cap R)$, the probability that a red jack is drawn from the deck.

There are **2** red jacks in the deck, so $P(J \cap R) = \dfrac{2}{52}$.

Step 3 Substitute the probabilities from Steps 1 and 2 into the formula for conditional probability.

$$P(J|R) = \frac{P(J \cap R)}{P(R)} = \frac{\frac{2}{52}}{\frac{4}{52}}$$

(handwritten) – divide the numerator by the reciprocal.

Step 4 Simplify the result.

$$P(J|R) = \frac{\frac{2}{52} \cdot 52}{\frac{4}{52} \cdot 52} = \frac{2}{4} = \frac{1}{2}$$

(handwritten) – you can also just write these for what they are

$$\frac{2}{4} = \frac{1}{2}$$

Your Turn

5. For a standard deck of playing cards, find the probability that a face card randomly drawn from the deck is a king. (The ace is *not* a face card.)

(handwritten) $P(F) = \dfrac{12}{52}$ $P(F \cap K) = \dfrac{4}{52} = \dfrac{\frac{4}{52}}{\frac{12}{52}} = \dfrac{4}{12} = \dfrac{1}{3}$

6. For a standard deck of playing cards, find the probability that a queen randomly drawn from the deck is a diamond.

(handwritten) $P(Q) = \dfrac{4}{52}$ $P(Q \cap D) = \dfrac{1}{52}$ $\dfrac{\frac{1}{52}}{\frac{4}{52}} = \dfrac{1}{4}$

7. When calculating a conditional probability from a two-way table, explain why it doesn't matter whether the table gives frequencies or relative frequencies.

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8. **Discussion** Is it possible to have $P(B|A) = P(A|B)$ for some events A and B? What conditions would need to exist?

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9. **Essential Question Check-In** In a two-way frequency table, suppose event A represents a row of the table and event B represents a column of the table. Describe how to find the conditional probability $P(A|B)$ using the frequencies in the table.

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In order to study the relationship between the amount of sleep a student gets and his or her school performance, a researcher collected data from 120 students. The two-way frequency table shows the number of students who passed and failed an exam and the number of students who got more or less than 6 hours of sleep the night before. Use the table to answer the questions in Exercises 1–3.

	Passed exam	Failed exam	Total
Less than 6 hours of sleep	12	10	22
More than 6 hours of sleep	90	8	98
Total	102	18	120

1. To the nearest percent, what is the probability that a student who failed the exam got less than 6 hours of sleep?

$$\frac{10}{18} = \frac{5}{9} \approx 56\%$$

- always look @ what was written first, that determines the total.

2. To the nearest percent, what is the probability that a <u>student who got less than 6 hours of sleep</u> failed the exam?

$$\frac{10}{22} = \frac{5}{11} \approx 45\%$$

- again look @ what was written first determines total.

3. To the nearest percent, what is the probability that a student got less than 6 hours of sleep and failed the exam?

— this is over the total

$$\frac{10}{120} = \frac{1}{12} = 8.3\%$$

4. You have a standard deck of playing cards from which you randomly select a card. Event D is getting a diamond, and event F is getting a face card (a jack, queen, or king).

Show that $P(D|F) = \dfrac{n(D \cap F)}{n(F)}$ and $P(D|F) = \dfrac{P(D \cap F)}{P(F)}$ are equal.

$\hookrightarrow \dfrac{3}{12} = \dfrac{1}{4}$ $\hookrightarrow \dfrac{\frac{3}{52}}{\frac{12}{52}} = \dfrac{3}{12} = \dfrac{1}{4}$

The table shows data in the previous table as relative frequencies (rounded to the nearest thousandth when necessary). Use the table for Exercises 5–7.

	Passed exam	Failed exam	Total
Less than 6 hours of sleep	0.100	0.083	0.183
More than 6 hours of sleep	0.750	0.067	0.817
Total	0.850	0.150	1.000

5. To the nearest percent, what is the probability that a student who passed the exam got more than 6 hours of sleep?

$$\frac{.750}{.850} \approx 0.882 = 88.2\%$$

6. To the nearest percent, what is the probability that a student who got more than 6 hours of sleep passed the exam?

$$\frac{.750}{.817} = .918 \approx 91.8\%$$

7. Which is greater, the probability that a student who got less than 6 hours of sleep passed the exam or the probability that a student who got more than 6 hours of sleep failed the exam? Explain.

$$\frac{.100}{.183} = .546 = 54.6\%$$ $$\frac{.067}{.817} = .082 \approx 8.2\%$$

You randomly draw a card from a standard deck of playing cards. Let *A* be the event that the card is an ace, let *B* be the event that the card is black, and let *C* be the event that the card is a club. Find the specified probability as a fraction.

(handwritten top right) 4 ACES / 2 black ACES / 13 clubs / 2 black aces / 4 aces

8. $P(A|B)$

$$\frac{P(A \cap B)}{P(B)} = \frac{1}{13}$$ $\frac{4\,aces}{52\,cards}$

9. $P(B|A)$

$$\frac{P(A \cap B)}{P(A)} = \frac{1}{2}$$ $\frac{2\,black\,aces}{4\,aces}$

10. $P(A|C)$

$$\frac{P(A \cap C)}{P(C)} = \frac{1}{13}$$ $\frac{1\,ace\text{-}club}{13\,clubs}$

11. $P(C|A)$

$$\frac{P(A \cap C)}{P(A)} = \frac{1}{4}$$ $\frac{1\,ace\,club}{4\,aces}$

12. $P(B|C)$

$$\frac{P(B \cap C)}{P(C)} = 1$$ $\frac{1\,black\,card}{1\,club}$

13. $P(C|B)$

$$\frac{P(B \cap C)}{P(B)} = \frac{1}{2}$$ $\frac{1\,club\,ace}{2\,black\,aces}$

14. A botanist studied the effect of a new fertilizer by choosing 100 orchids and giving 70% of these plants the fertilizer. Of the plants that got the fertilizer, 40% produced flowers within a month. Of the plants that did not get the fertilizer, 10% produced flowers within a month.

a. Use the given information to complete the two-way frequency table.

	Received fertilizer	Did not receive fertilizer	Total
Did not flower in one month	70-28 = **42**	30-3 **27**	**69**
Flowered in one month	.4(70) **28**	.1(30) **3**	**31**
Total	**70**	**30**	**100**

b. To the nearest percent, what is the probability that an orchid that produced flowers got fertilizer?

$$\frac{28}{31} = 90.3\%$$

c. To the nearest percent, what is the probability that an orchid that got fertilizer produced flowers?

$$\frac{28}{70} = \frac{2}{5} = 40\%$$

15. At a school fair, a box contains 24 yellow balls and 76 red balls. One-fourth of the balls of each color are labeled "Win a prize." Match each description of a probability with its value as a percent.

A. The probability that a randomly selected ball labeled "Win a prize" is yellow

B. The probability that a randomly selected ball labeled "Win a prize" is red

C. The probability that a randomly selected ball is labeled "Win a prize" and is red

D. The probability that a randomly selected yellow ball is labeled "Win a prize"

B 76%

D 25%

A 24%

C 19%

16. A teacher gave her students two tests. If 45% of the students passed both tests and 60% passed the first test, what is the probability that a student who passed the first test also passed the second?

$$\frac{.45}{.60} = .75 \approx 75\%$$

17. You randomly select two marbles, one at a time, from a pouch containing blue and green marbles. The probability of selecting a blue marble on the first draw and a green marble on the second draw is 25%, and the probability of selecting a blue marble on the first draw is 56%. To the nearest percent, what is the probability of selecting a green marble on the second draw, given that the first marble was blue?

$$\frac{.25}{.56} \approx .45 = 45\%$$

You roll two number cubes, one red and one blue. The table shows the probabilities for events based on whether or not a 1 is rolled on each number cube. Use the table to find the specified conditional probability, expressed as a fraction. Then show that the conditional probability is correct by listing the possible outcomes as ordered pairs of the form (number on red cube, number on blue cube) and identifying the successful outcomes.

	Rolling a 1 on the red cube	Not rolling a 1 on the red cube	Total
Rolling a 1 on the blue cube	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{1}{6}$
Not rolling a 1 on the blue cube	$\frac{5}{36}$	$\frac{25}{36}$	$\frac{5}{6}$
Total	$\frac{1}{6}$	$\frac{5}{6}$	1

18. P(not rolling a 1 on the blue cube | rolling a 1 on the red cube)

$$\frac{\frac{5}{36}}{\frac{1}{6}} = \frac{5}{36} \cdot \frac{6}{1} = \frac{5}{6}$$

19. P(not rolling a 1 on the blue cube | not rolling a 1 on the red cube)

$$\frac{\frac{25}{36}}{\frac{5}{6}} \quad \frac{25}{36} \cdot \frac{6}{5} = \frac{5}{6}$$

20. The table shows the results of a quality-control study at a computer factory.

|	Shipped	Not shipped	Total
Defective	3	7	10
Not defective	89	1	90
Total	92	8	100

a. To the nearest tenth of a percent, what is the probability that a shipped computer is not defective?

b. To the nearest tenth of a percent, what is the probability that a defective computer is shipped?

a) $\dfrac{89}{92} \approx .967 \approx 96.7\%$

b) $\dfrac{3}{10} \approx .3 \approx 30\%$

H.O.T. Focus on Higher Order Thinking

21. Analyze Relationships In the Venn diagram, the circles representing events A and B divide the sample space S into four regions: the overlap of the circles, the part of A not in the overlap, the part of B not in the overlap, and the part of S not in A or B. Suppose that the area of each region is proportional to the number of outcomes that fall within the region. Which conditional probability is greater: $P(A|B)$ or $P(B|A)$? Explain.
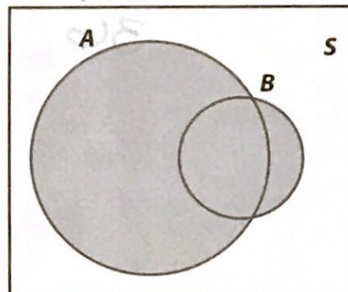
22. Explain the Error A student was asked to use the table shown to find the probability, to nearest percent, that a participant in a study of a new medicine for migraine headaches did not take the medicine, given that the participant reported no headaches.

	Took medicine	No medicine	Total
Headache	11	13	24
No headache	54	22	76
Total	65	35	100

The student made the following calculation.

$P(\text{no medicine} \mid \text{no headache}) = \frac{22}{35} \approx 0.63 = 63\%$

Explain the student's error, and find the correct probability.

- no headache before no medicine

- should be $\frac{22}{76} = \frac{11}{38} = .29 \approx 29\%$

23. Communicate Mathematical Ideas Explain how a conditional probability based on a two-way frequency table effectively reduces it to a one-way table. In your explanation, refer to the two-way table shown, which lists frequencies for events A, B, and their complements. Highlight the part of the table that supports your explanation.

	A	Not A	Total
B	$n(A \cap B)$	$n(\text{not } A \cap B)$	$n(B)$
Not B	$n(A \cap \text{not } B)$	$n(\text{not } A \cap \text{not } B)$	$n(\text{not } B)$
Total	$n(A)$	$n(\text{not } A)$	$n(S)$