Name ______ Class ______ Date ______

22.1 Conditional Probability

Essential Question: How do you calculate a conditional probability?

Resource Locker

Explore 1 Finding Conditional Probabilities from a Two-Way Frequency Table

The probability that event A occurs given that event B has already occurred is called the **conditional probability** of A given B and is written $P(A|B)$.

One hundred migraine headache sufferers participated in a study of a new medicine. Some were given the new medicine, and others were not. After one week, participants were asked if they had experienced a headache during the week. The two-way frequency table shows the results.

	Took medicine	No medicine	Total
Headache	11	13	24
No headache	54	22	76
Total	65	35	100

probabilities are either fractions or percents

Let event A be the event that a participant did not get a headache. Let event B be the event that a participant took the medicine.

(A) To the nearest percent, what is the probability that a participant who took the medicine did not get a headache?

65 participants took the medicine. So, $P(A|B) = \frac{54}{65} \approx 83\%$.

Of these, 54 did not get a headache.

(B) To the nearest percent, what is the probability that a participant who did not get a headache took the medicine?

76 participants did not get a headache. So, $P(B|A) = \frac{54}{76} \approx 71\%$.

Of these, 54 took the medicine.

(C) Let $n(A)$ be the number of participants who did not get a headache, $n(B)$ be the number of participants who took the medicine, and $n(A \cap B)$ be the number of participants who took the medicine and did not get a headache.

$n(A) = 76$ $n(B) = 65$ $n(A \cap B) = 76$

Express $P(A|B)$ and $P(B|A)$ in terms of $n(A)$, $n(B)$, and $n(A \cap B)$.

B has to occur before A can happen $P(A|B) = \frac{n(A \cap B)}{n(B)}$ $P(B|A) = \frac{n(A \cap B)}{n(A)}$

means that A has to occur before B can happen

Reflect

1. For the question "What is the probability that a participant who did not get a headache took the medicine?", what event is assumed to have already occurred?

2. In general, does it appear that $P(A|B) = P(B|A)$? Why or why not?

Explore 2 Finding Conditional Probabilities from a Two-Way Relative Frequency Table

You can develop a formula for $P(A|B)$ that uses relative frequencies (which are probabilities) rather than frequencies (which are counts).

	Took medicine	No medicine	Total
Headache	11	13	24
No headache	54	22	76
Total	65	35	100

Ⓐ To obtain relative frequencies, divide every number in the table by 100, the total number of participants in the study.

percentages always add up to 100%, and if use decimals has to add to 1.

	Took medicine	No medicine	Total
Headache	0.11	0.13	0.24
No headache	0.54	0.22	0.76
Total	0.65	0.35	1

Ⓑ Recall that event A is the event that a participant did not get a headache and that event B is the event that a participant took the medicine. Use the relative frequency table from Step A to find $P(A)$, $P(B)$, and $P(A \cap B)$.

$P(A) = 0.76$; $P(B) = 0.65$

$P(A \cap B) = 0.54$

Ⓒ In the first Explore, you found the conditional probabilities $P(A|B) \approx 83\%$ and $P(B|A) \approx 71\%$ by using the frequencies in the two-way frequency table. Use the relative frequencies from the table in Step A to find the equivalent conditional probabilities.

$P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\boxed{0.54}}{\boxed{0.65}} \approx \boxed{83}\%$ $\qquad$ $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{\boxed{0.54}}{\boxed{0.76}} \approx \boxed{71}\%$

(D) Generalize the results by using $n(S)$ as the number of elements in the sample space (in this case, the number of participants in the study). For instance, you can write $P(A) = \frac{n(A)}{n(S)}$. Write each of the following probabilities in a similar way.

$P(B) = \frac{n(B)}{n(S)}$ $\quad$ $P(A \cap B) = \frac{n(A \cap B)}{n(S)}$ $\quad$ $P(A \mid B) = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = \frac{P(A \cap B)}{P(B)}$

Reflect

3. Why are the two forms of $P(A \cap B)$, $\frac{n(A \cap B)}{n(B)}$ and $\frac{P(A \cap B)}{P(B)}$, equivalent?

4. What is a formula for $P(B \mid A)$ that involves probabilities rather than counts? How do you obtain this formula from the fact that $P(B \mid A) = \frac{n(A \cap B)}{n(A)}$?

Explain 1 Using the Conditional Probability Formula

In the previous Explore, you discovered the following formula for conditional probability.

Conditional Probability

The conditional probability of A given B (that is, the probability that event A occurs given that event B occurs) is as follows:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

need to know and then swap (B|A) when needed.

Example 1 Find the specified probability.

(A) For a standard deck of playing cards, find the probability that a red card randomly drawn from the deck is a jack.

Step 1 Find $P(R)$, the probability that a red card is drawn from the deck.

There are 26 red cards in the deck of 52 cards, so $P(R) = \frac{26}{52}$.

Step 2 Find $P(J \cap R)$, the probability that a red jack is drawn from the deck.

There are 2 red jacks in the deck, so $P(J \cap R) = \frac{2}{52}$.

Step 3 Substitute the probabilities from Steps 1 and 2 into the formula for conditional probability.

$$P(J|R) = \frac{P(J \cap R)}{P(R)} = \frac{\frac{2}{52}}{\frac{26}{52}}$$

Step 4 Simplify the result.

$$P(J|R) = \frac{\frac{2}{52} \cdot 52}{\frac{26}{52} \cdot 52} = \frac{2}{26} = \frac{1}{13}$$

(B) For a standard deck of playing cards, find the probability that a jack randomly drawn from the deck is a red card.

Step 1 Find $P(J)$, the probability that a jack is drawn from the deck.

There are 4 jacks in the deck of 52 cards, so $P(J) = \frac{4}{52}$.

Step 2 Find $P(J \cap R)$, the probability that a red jack is drawn from the deck.

There are 2 red jacks in the deck, so $P(J \cap R) = \frac{2}{52}$.

Step 3 Substitute the probabilities from Steps 1 and 2 into the formula for conditional probability.

$$P(R|J) = \frac{P(J \cap R)}{P(J)} = \frac{\frac{2}{52}}{\frac{4}{52}}$$

Step 4 Simplify the result.

$$P(R|J) = \frac{\frac{2}{52} \cdot 52}{\frac{4}{52} \cdot 52} = \frac{2}{4} = \frac{1}{2}$$

Your Turn

5. For a standard deck of playing cards, find the probability that a face card randomly drawn from the deck is a king. (The ace is *not* a face card.)

$P(F) = \frac{12}{52}$

$P(F \cap K) = \frac{4}{52}$

$$\frac{\frac{4}{52}}{\frac{12}{52}} = \frac{4}{52} \cdot \frac{52}{12} = \frac{4}{12} = \frac{1}{3}$$

6. For a standard deck of playing cards, find the probability that a queen randomly drawn from the deck is a diamond.

$P(Q) = \frac{4}{52}$

$P(Q \cap D) = \frac{1}{52}$

$$\frac{\frac{1}{52}}{\frac{4}{52}} = \frac{1}{52} \cdot \frac{52}{4} = \frac{1}{4}$$

Hebrews 10:22-25

Elaborate

7. When calculating a conditional probability from a two-way table, explain why it doesn't matter whether the table gives frequencies or relative frequencies.

8. **Discussion** Is it possible to have $P(B|A) = P(A|B)$ for some events A and B? What conditions would need to exist?

9. **Essential Question Check-In** In a two-way frequency table, suppose event A represents a row of the table and event B represents a column of the table. Describe how to find the conditional probability $P(A|B)$ using the frequencies in the table.

Evaluate: Homework and Practice

Personal Math Trainer
- Online Homework
- Hints and Help
- Extra Practice

In order to study the relationship between the amount of sleep a student gets and his or her school performance, a researcher collected data from 120 students. The two-way frequency table shows the number of students who passed and failed an exam and the number of students who got more or less than 6 hours of sleep the night before. Use the table to answer the questions in Exercises 1–3.

	Passed exam	Failed exam	Total
Less than 6 hours of sleep	12	10	22
More than 6 hours of sleep	90	8	98
Total	102	18	120

1. To the nearest percent, what is the probability that a student who failed the exam got less than 6 hours of sleep?

$P(L|F)$ $\frac{n(L\cap F)}{n(F)} = \frac{10}{18} = \frac{5}{9} \approx 56\%$

* who means total for that row or column *

2. To the nearest percent, what is the probability that a student who got less than 6 hours of sleep failed the exam?

$P(F|L)$ $\frac{n(L\cap F)}{n(L)} = \frac{10}{22} = \frac{5}{11} \approx 45\%$

3. To the nearest percent, what is the probability that a student got less than 6 hours of sleep and failed the exam?

$P(L \cap F) = \frac{n(L \cap F)}{n(S)} = \frac{10}{120} \approx 8\%$

* got means the overall total *

4. You have a standard deck of playing cards from which you randomly select a card. Event D is getting a diamond, and event F is getting a face card (a jack, queen, or king).

Show that $P(D|F) = \frac{n(D \cap F)}{n(F)}$ and $P(D|F) = \frac{P(D \cap F)}{P(F)}$ are equal.

$P(D|F) = \frac{n(D \cap F)}{n(F)} = \frac{3}{12} = \frac{1}{4}$ $P(D|F) = \frac{P(D \cap F)}{P(F)} = \frac{\frac{3}{52}}{\frac{12}{52}} = \frac{3}{12} = \frac{1}{4}$

The table shows data in the previous table as relative frequencies (rounded to the nearest thousandth when necessary). Use the table for Exercises 5–7.

	Passed exam	Failed exam	Total
Less than 6 hours of sleep	0.100	0.083	0.183
More than 6 hours of sleep	0.750	0.067	0.817
Total	0.850	0.150	1.000

5. To the nearest percent, what is the probability that a student who passed the exam got more than 6 hours of sleep?

$P(M|Pa) = \frac{P(M \cap Pa)}{P(Pa)} = \frac{0.750}{0.850} \approx 0.882 \approx 88\%$

6. To the nearest percent, what is the probability that a student who got more than 6 hours of sleep passed the exam?

$P(Pa|M) = \frac{n(M \cap Pa)}{n(M} = \frac{0.750}{0.817} \approx 0.918 \approx 92\%$

7. Which is greater, the probability that a student who got less than 6 hours of sleep passed the exam or the probability that a student who got more than 6 hours of sleep failed the exam? Explain.

$P(Pa|L) = \frac{P(L \cap Pa)}{P(L)} = \frac{0.100}{0.183} \approx 0.546 \approx 55\%$

$P(F|M) = \frac{P(M \cap F)}{P(M)} = \frac{0.067}{0.817} = 0.082 = 8\%$

- so that the probability a student who got less than 6 hours of sleep passed the exam is greater.

You randomly draw a card from a standard deck of playing cards. Let A be the event that the card is an ace, let B be the event that the card is black, and let C be the event that the card is a club. Find the specified probability as a fraction.

8. $P(A|B)$

$\frac{P(A \cap B)}{P(B)} = \frac{1}{13}$

9. $P(B|A)$

$\frac{P(A \cap B)}{P(A)} = \frac{1}{2}$

10. $P(A|C)$

$\frac{P(A \cap C)}{P(C)} = \frac{1}{13}$

11. $P(C|A)$

$\frac{P(A \cap C)}{P(A)} = \frac{1}{4}$

12. $P(B|C)$

$\frac{P(B \cap C)}{P(C)} = 1$

13. $P(C|B)$

$\frac{P(B \cap C)}{P(B)} = \frac{1}{2}$

14. A botanist studied the effect of a new fertilizer by choosing 100 orchids and giving 70% of these plants the fertilizer. Of the plants that got the fertilizer, 40% produced flowers within a month. Of the plants that did not get the fertilizer, 10% produced flowers within a month.

a. Use the given information to complete the two-way frequency table.

	Received fertilizer	Did not receive fertilizer	Total
Did not flower in one month	70-28 42	30-3 27	69
Flowered in one month	0.4(70) 28	0.1(30) 3	31
Total	0.7(100) 70	0.3(100) 30	100

b. To the nearest percent, what is the probability that an orchid that produced flowers got fertilizer?

$P(Fe|Fl) = \frac{n(Fe \cap Fl)}{n(Fl)} = \frac{28}{31} \approx 90\%$

c. To the nearest percent, what is the probability that an orchid that got fertilizer produced flowers?

$P(Fl|Fe) = \frac{n(Fe \cap Fl)}{n(Fe)} = \frac{28}{70} = \frac{2}{5} = 40\%$

15. At a school fair, a box contains 24 yellow balls and 76 red balls. One-fourth of the balls of each color are labeled "Win a prize." Match each description of a probability with its value as a percent.

A. The probability that a randomly selected ball labeled "Win a prize" is yellow — B 76%

B. The probability that a randomly selected ball labeled "Win a prize" is red — D 25%

C. The probability that a randomly selected ball is labeled "Win a prize" and is red — A 24%

D. The probability that a randomly selected yellow ball is labeled "Win a prize" — C 19%

16. A teacher gave her students two tests. If 45% of the students passed both tests and 60% passed the first test, what is the probability that a student who passed the first test also passed the second?

$$P(T_2|T_1) = \frac{P(T_1 \cap T_2)}{P(T_1)} = \frac{0.45}{0.60} \approx 0.75 \approx 75\%$$

17. You randomly select two marbles, one at a time, from a pouch containing blue and green marbles. The probability of selecting a blue marble on the first draw and a green marble on the second draw is 25%, and the probability of selecting a blue marble on the first draw is 56%. To the nearest percent, what is the probability of selecting a green marble on the second draw, given that the first marble was blue?

$$P(G|B) = \frac{P(B \cap G)}{P(B)} = \frac{0.25}{0.56} \approx 0.45 = 45\%$$