Name ______________________ Class ____________ Date ________

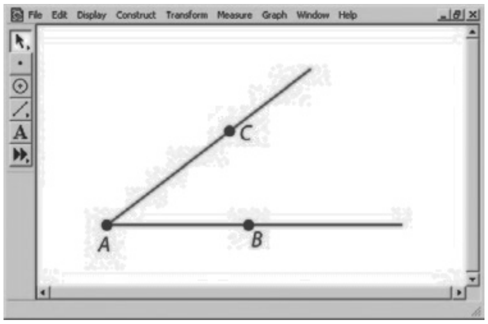
13.2 Sine and Cosine Ratios

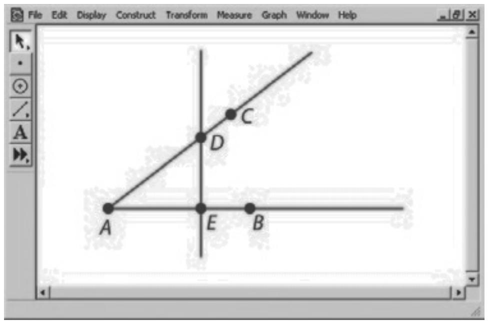
Essential Question: How can you use the sine and cosine ratios, and their inverses, in calculations involving right triangles?

Resource Locker

Explore — Investigating Ratios in a Right Triangle

You can use geometry software or an online tool to explore ratios of side lengths in right triangles.





(A) Construct three points A, B, and C.
Construct rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Move C so that $\angle A$ is acute.

(B) Construct point D on $\overline{AC}$. Construct a line through D perpendicular to $\overline{AB}$. Construct point E as the intersection of the perpendicular line and $\overline{AB}$.

(C) Measure $\angle A$. Measure the side lengths DE, AE, and AD of $\triangle ADE$.

(D) Calculate the ratios $\frac{DE}{AD}$ and $\frac{AE}{AD}$.

Reflect

1. Drag D along $\overrightarrow{AC}$. What happens to $m\angle A$ as D moves along $\overrightarrow{AC}$? What postulate or theorem guarantees that the different triangles formed are similar to each other?

2. As you move D along $\overrightarrow{AC}$, what happens to the values of the ratios $\frac{DE}{AD}$ and $\frac{AE}{AD}$? Use the properties of similar triangles to explain this result.

3. Move C. What happens to $m\angle A$? With a new value of $m\angle A$, note the values of the two ratios. What happens to the ratios if you drag D along $\overrightarrow{AC}$?

Explain 1 Finding the Sine and Cosine of an Angle

Trigonometric Ratios

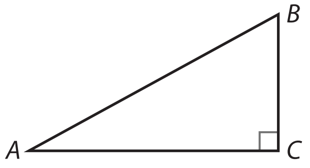
A **trigonometric ratio** is a ratio of two sides of a right triangle. You have already seen one trigonometric ratio, the tangent. There are two additional trigonometric ratios, the sine and the cosine, that involve the hypotenuse of a right triangle.

The **sine** of $\angle A$, written $\sin A$, is defined as follows:

$$\sin A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}} = \frac{BC}{AB}$$

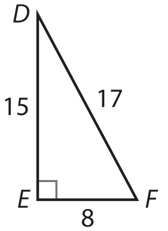
The **cosine** of $\angle A$, written $\cos A$, is defined as follows:

$$\cos A = \frac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{AC}{AB}$$



You can use these definitions to calculate trigonometric ratios.

Example 1 **Write sine and cosine of each angle as a fraction and as a decimal rounded to the nearest thousandth.**



(A) $\angle D$

$$\sin D = \frac{\text{length of leg opposite } \angle D}{\text{length of hypotenuse}} = \frac{EF}{DF} = \frac{8}{17} \approx 0.471$$

$$\cos D = \frac{\text{length of leg adjacent to } \angle D}{\text{length of hypotenuse}} = \frac{DE}{DF} = \frac{15}{17} \approx 0.882$$

(B) $\angle F$

$$\sin F = \frac{\text{length of leg opposite to } \angle F}{\text{length of hypotenuse}} = \frac{DE}{DF} = \frac{\square}{\square} \approx \square$$

$$\cos F = \frac{\text{length of leg adjacent to } \angle F}{\text{length of hypotenuse}} = \frac{\square}{\square} \approx \square$$

Reflect

4. What do you notice about the sines and cosines you found? Do you think this relationship will be true for any pair of acute angles in a right triangle? Explain.

5. In a right triangle $\triangle PQR$ with $PR = 5$, $QR = 3$, and $m\angle Q = 90°$, what are the values of $\sin P$ and $\cos P$?

$\sin P = \square$

$\cos P = \square$

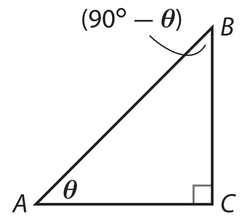
Explain 2 Using Complementary Angles

The acute angles of a right triangle are complementary. Their trigonometric ratios are related to each other as shown in the following relationship.

Trigonometric Ratios of Complementary Angles

If $\angle A$ and $\angle B$ are the acute angles in a right triangle, then $\sin A = \cos B$ and $\cos A = \sin B$.

Therefore, if θ ("theta") is the measure of an acute angle, then $\sin \theta = \cos\left(90° - \theta\right)$ and $\cos \theta = \sin\left(90° - \theta\right)$.



You can use these relationships to write equivalent expressions.

Example 2 **Write each trigonometric expression.**

(A) Given that $\sin 38° \approx 0.616$, write the cosine of a complementary angle in terms of the sine of 38°. Then find the cosine of the complementary angle.

Use an expression relating trigonometric ratios of complementary angles.

$$\sin \theta = \cos(90° - \theta)$$

Substitute 38 into both sides. $\sin 38° = \cos(90° - 38°)$

Simplify. $\sin 38° = \cos 52°$

Substitute for sin 38°. $0.616 \approx \cos 52°$

So, the cosine of the complementary angle is about 0.616.

(B) Given that $\cos 60° = 0.5$, write the sine of a complementary angle in terms of the cosine of 60°. Then find the sine of the complementary angle.

Use an expression relating trigonometric ratios of complementary angles.

$$\cos \theta = \sin\left(90° - \theta\right)$$

Substitute $\boxed{}$ into both sides. $\cos \boxed{}° = \sin\left(90° - \boxed{}°\right)$

Simplify the right side. $\cos \boxed{}° = \sin \boxed{}°$

Substitute for the cosine of $\boxed{}°$. $\boxed{} = \sin \boxed{}°$

So, the sine of the complementary angle is 0.5.

Reflect

6. What can you conclude about the sine and cosine of 45°? Explain.

7. **Discussion** Is it possible for the sine or cosine of an acute angle to equal 1? Explain.

Your Turn

Write each trigonometric expression.

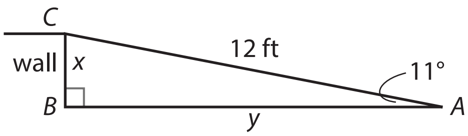
8. Given that $\cos 73° \approx 0.292$, write the sine of a complementary angle.

9. Given that $\sin 45° \approx 0.707$, write the cosine of a complementary angle.

Explain 3 Finding Side Lengths using Sine and Cosine

You can use sine and cosine to solve real-world problems.

Example 3 **A 12-ft ramp is installed alongside some steps to provide wheelchair access to a library. The ramp makes an angle of 11° with the ground. Find each dimension, to the nearest tenth of a foot.**



(A) Find the height x of the wall.

Use the definition of sine.	$\sin A = \dfrac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}} = \dfrac{AB}{AC}$
Substitute 11° for A, x for BC, and 12 for AC.	$\sin 11° = \dfrac{x}{12}$
Multiply both sides by 12.	$12 \sin 11° = x$
Use a calculator to evaluate the expression.	$x \approx 2.3$

So, the height of the wall is about 2.3 feet.

(B) Find the distance y that the ramp extends in front of the wall.

Use the definition of cosine. $\cos A = \dfrac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}} = \dfrac{AB}{AC}$

Substitute $\square°$ for A, y for AB, and $\square$ for AC. $\cos \square° = \dfrac{y}{\square}$

Multiply both sides by $\square$. $\square \cos \square° = y$

Use a calculator to evaluate the expression. $y \approx \square$

So, the ramp extends in front of the wall about $\square$ feet.

Reflect

10. Could you find the height of the wall using the cosine? Explain.

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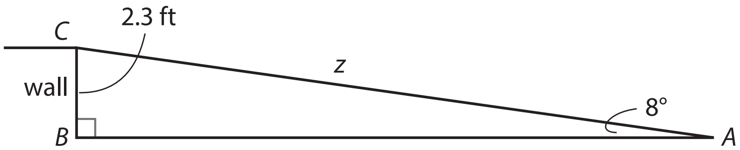
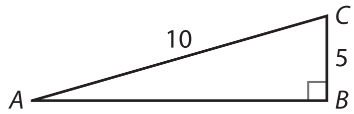
Your Turn

11. Suppose a new regulation states that the maximum angle of a ramp for wheelchairs is 8°. At least how long must the new ramp be? Round to the nearest tenth of a foot.



Explain 4 Finding Angle Measures using Sine and Cosine

In the triangle, $\sin A = \frac{5}{10} = \frac{1}{2}$. However, you already know that $\sin 30° = \frac{1}{2}$. So you can conclude that $m\angle A = 30°$,
and write $\sin^{-1}\left(\frac{1}{2}\right) = 30°$.



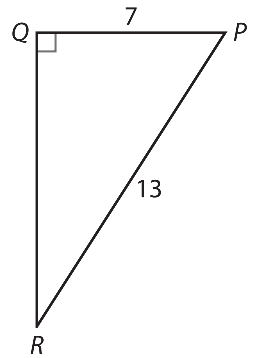
Extending this idea, the **inverse trigonometric ratios** for sine and cosine are defined as follows:

Given an acute angle, $\angle A$,

- if $\sin A = x$, then $\sin^{-1} x = m\angle A$, read as "inverse sine of x"
- if $\cos A = x$, then $\cos^{-1} x = m\angle A$, read as "inverse cosine of x"

You can use a calculator to evaluate inverse trigonometric expressions.

Example 4 **Find the acute angle measures in $\triangle PQR$, to the nearest degree.**



(A) Write a trigonometric ratio for $\angle R$.

Since the lengths of the hypotenuse and the opposite leg are given, use the sine ratio. $\sin R = \frac{PQ}{PR}$

Substitute 7 for PQ and 13 for PR. $\sin R = \frac{7}{13}$

(B) Write and evaluate an inverse trigonometric ratio to find $m\angle R$ and $m\angle P$.

Start with the trigonometric ratio for $\angle R$. $\sin R = \square$

Use the definition of the inverse sine ratio. $m\angle R = \sin^{-1} \square$

Use a calculator to evaluate the inverse sine ratio. $m\angle R = \square^\circ$

Write a cosine ratio for $\angle P$. $\cos P = \frac{PQ}{PR}$

Substitute $\square$ for PQ and $\square$ for PR. $\cos P = \square$

Use the definition of the inverse cosine ratio. $m\angle P = \cos^{-1} \square$

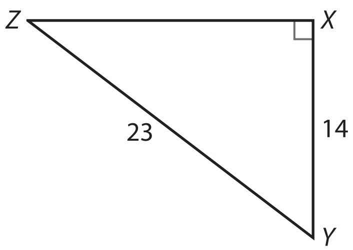
Use a calculator to evaluate the inverse cosine ratio. $m\angle P = \square^\circ$

Reflect

12. How else could you have determined $m\angle P$?

Your Turn

Find the acute angle measures in $\triangle XYZ$, to the nearest degree.

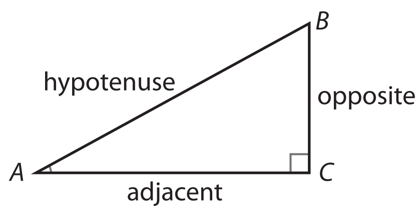


13. $m\angle Y$

14. $m\angle Z$

Elaborate

15. How are the sine and cosine ratios for an acute angle of a right triangle defined?



16. How are the inverse sine and cosine ratios for an acute angle of a right triangle defined?

17. **Essential Question Check-In** How do you find an unknown angle measure in a right triangle?

Evaluate: Homework and Practice

Personal Math Trainer

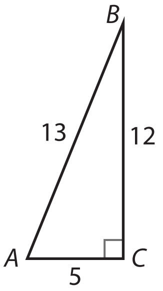
- Online Homework
- Hints and Help
- Extra Practice

Write each trigonometric expression. Round trigonometric ratios to the nearest thousandth.

1. Given that $\sin 60° \approx 0.866$, write the cosine of a complementary angle.

2. Given that $\cos 26° \approx 0.899$, write the sine of a complementary angle.

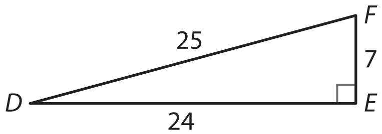
Write each trigonometric ratio as a fraction and as a decimal, rounded (if necessary) to the nearest thousandth.



3. $\sin A$

4. $\cos A$

5. $\cos B$



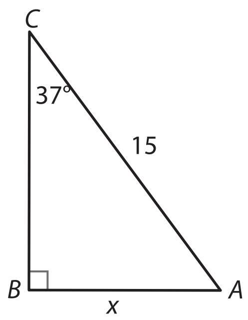
6. $\sin D$

7. $\cos F$

8. $\sin F$

Find the unknown length *x* in each right triangle, to the nearest tenth.

9.



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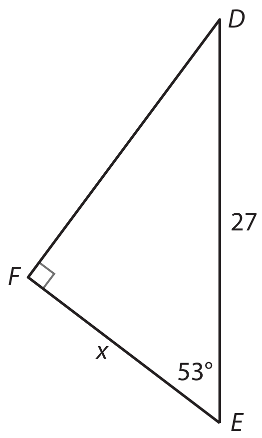

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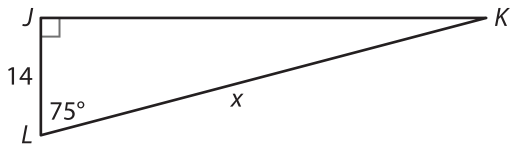

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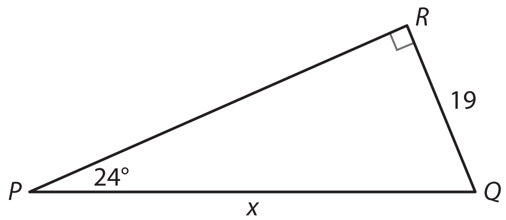
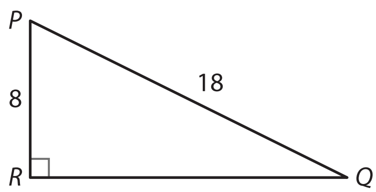
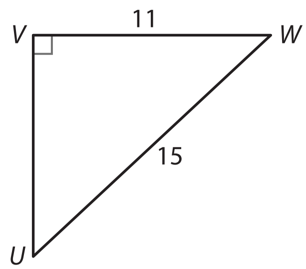


Find each acute angle measure, to the nearest degree.





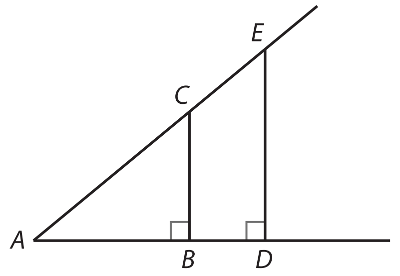
13. m$\angle P$

14. m$\angle Q$

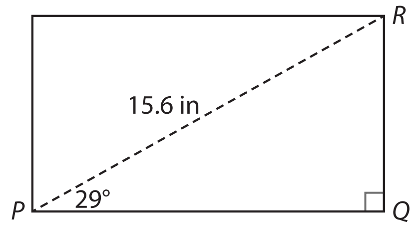
15. m$\angle U$

16. m$\angle W$

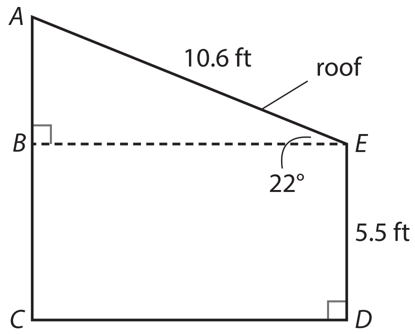
17. Use the property that corresponding sides of similar triangles are proportional to explain why the trigonometric ratio sin A is the same when calculated in $\triangle ADE$ as in $\triangle ABC$.



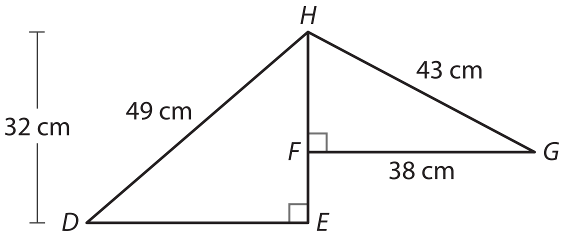
18. **Technology** The specifications for a laptop computer describe its screen as measuring 15.6 in. However, this is actually the length of a diagonal of the rectangular screen, as represented in the figure. How wide is the screen horizontally, to the nearest tenth of an inch?



19. **Building** Sharla's bedroom is directly under the roof of her house. Given the dimensions shown, how high is the ceiling at its highest point, to the nearest tenth of a foot?

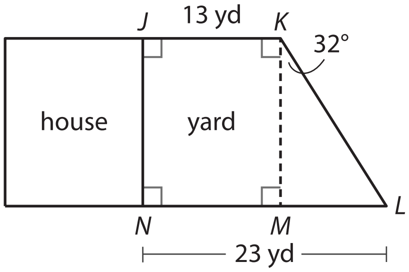


20. **Zoology** You can sometimes see an eagle gliding with its wings flexed in a characteristic double-vee shape. Each wing can be modeled as two right triangles as shown in the figure. Find the measure of the angle in the middle of the wing, $\angle DHG$ to the nearest degree.

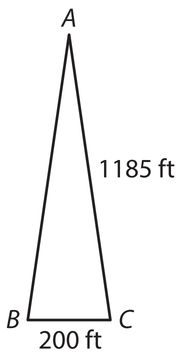


21. **Algebra** Find a pair of acute angles that satisfy the equation $\sin(3x + 9) = \cos(x + 5)$. Check that your answers make sense.

22. **Multi-Step** Reginald is planning to fence his back yard. Every side of the yard except for the side along the house is to be fenced, and fencing costs \$3.50/yd. How much will the fencing cost?



23. **Architecture** The sides of One World Trade Center in New York City form eight isosceles triangles, four of which are 200 ft long at their base *BC*. The length *AC* of each sloping side is approximately 1185 ft.



Find the measure of the apex angle *BAC* of each isosceles triangle, to the nearest tenth of a degree. (*Hint:* Use the midpoint *D* of $\overline{BC}$ to create two right triangles.)

H.O.T. Focus on Higher Order Thinking

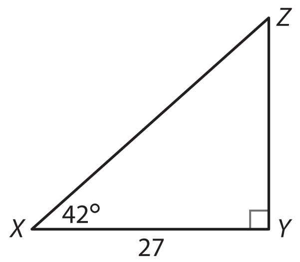
24. Explain the Error Melissa has calculated the length of $\overline{XZ}$ in $\triangle XYZ$. Explain why Melissa's answer must be incorrect, and identify and correct her error.

Melissa's solution:

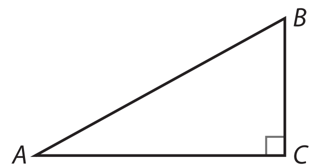
$$\cos X = \frac{XZ}{XY}$$

$$XZ = \frac{XY}{\cos X}$$

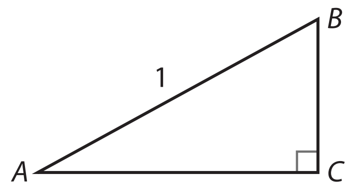
$$XZ = 27 \cos 42° \approx 20.1$$



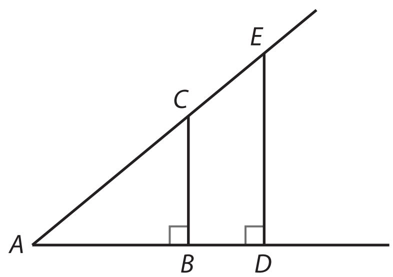
25. Communicate Mathematical Ideas Explain why the sine and cosine of an acute angle are always between 0 and 1.



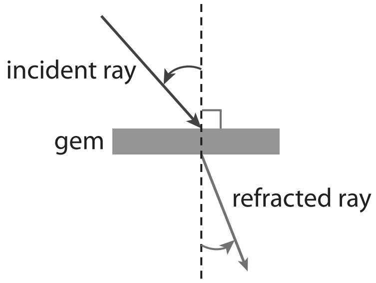
26. Look for a Pattern In $\triangle ABC$, the hypotenuse $\overline{AB}$ has a length of 1. Use the Pythagorean Theorem to explore the relationship between the squares of the sine and cosine of $\angle A$, written $\sin^2 A$ and $\cos^2 A$. Could you derive this relationship using a right triangle without any lengths specified? Explain.



27. Justify Reasoning Use the Triangle Proportionality Theorem to explain why the trigonometric ratio cos A is the same when calculated in $\triangle ADE$ as in $\triangle ABC$.



Lesson Performance Task



As light passes from a vacuum into another medium, it is *refracted*—that is, its direction changes. The ratio of the sine of the angle of the incoming *incident* ray, I, to the sine of the angle of the outgoing *refracted* ray, r, is called the *index of refraction*:

$n = \frac{\sin I}{\sin r}$. where n is the index of refraction.

This relationship is important in many fields, including gemology, the study of precious stones. A gemologist can place an unidentified gem into an instrument called a refractometer, direct an incident ray of light at a particular angle into the stone, measure the angle of the refracted ray, and calculate the index of refraction. Because the indices of refraction of thousands of gems are known, the gemologist can then identify the gem.

Gem	Index of Refraction
Hematite	2.94
Diamond	2.42
Zircon	1.95
Azurite	1.85
Sapphire	1.77
Tourmaline	1.62
Serpentine	1.56
Coral	1.49
Opal	1.39

1. Identify the gem, given these angles obtained from a refractometer:

a. $I = 71°, r = 29°$

b. $I = 51°, r = 34°$

c. $I = 45°, r = 17°$

2. A thin slice of sapphire is placed in a refractometer. The angle of the incident ray is 56°. Find the angle of the refracted ray to the nearest degree.

3. An incident ray of light struck a slice of serpentine. The resulting angle of refraction measured 21°. Find the angle of incidence to the nearest degree.

4. Describe the error(s) in a student's solution and explain why they were error(s):

$$n = \frac{\sin I}{\sin r}$$
$$= \frac{\sin 51°}{\sin 34°}$$
$$= \frac{51°}{34°}$$
$$= 1.5 \rightarrow \text{coral}$$