Name ____________________ Class __________ Date ______

13.4 Problem Solving with Trigonometry

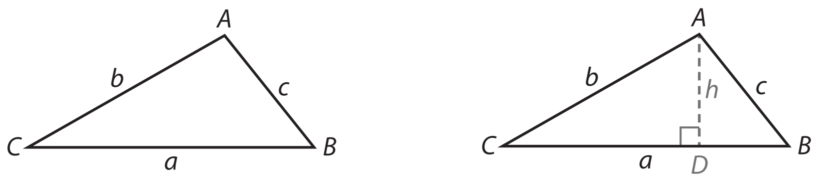
Essential Question: How can you solve a right triangle?

Resource Locker

Explore Deriving an Area Formula

You can use trigonometry to find the area of a triangle without knowing its height.

(A) Suppose you draw an altitude $\overline{AD}$ to side $\overline{BC}$ of $\triangle ABC$. Then write an equation using a trigonometric ratio in terms of $\angle C$, the height h of $\triangle ABC$, and the length of one of its sides.



(B) Solve your equation from Step A for h.

(C) Complete this formula for the area of $\triangle ABC$ in terms of h and another of its side lengths: Area $= \frac{1}{2}\boxed{}$

(D) Substitute your expression for h from Step B into your formula from Step C.

Reflect

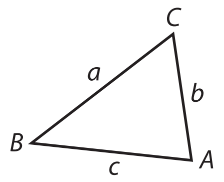
1. Does the area formula you found work if $\angle C$ is a right angle? Explain.

2. Suppose you used a trigonometric ratio in terms of $\angle B$, h, and a different side length. How would this change your findings? What does this tell you about the choice of sides and included angle?

Explain 1 Using the Area Formula

Area Formula for a Triangle in Terms of its Side Lengths

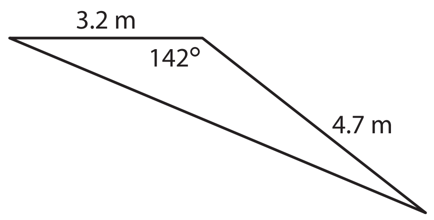
The area of $\triangle ABC$ with sides a, b, and c can be found using the lengths of two of its sides and the sine of the included angle: Area $= \frac{1}{2}bc \sin A$, Area $= \frac{1}{2}ac \sin B$, or Area $= \frac{1}{2}\, ab \sin C$.



You can use any form of the area formula to find the area of a triangle, given two side lengths and the measure of the included angle.

Example 1 **Find the area of each triangle to the nearest tenth.**

(A)



Let the known side lengths be a and b.	$a = 3.2$ m and $b = 4.7$ m
Let the known angle be $\angle C$.	$m\angle C = 142°$
Substitute in the formula Area $= \frac{1}{2}ab \sin C$.	Area $= \frac{1}{2}(3.2)(4.7)\sin 142°$
Evaluate, rounding to the nearest tenth.	Area ≈ 4.6 m^2

Ⓑ In $\triangle DEF$, $DE = 9$ in., $DF = 13$ in., and $m\angle D = 57°$.

Sketch $\triangle DEF$ and check that $\angle D$ is the included angle.

Write the area formula in terms of $\triangle DEF$. $\quad$ Area $= \frac{1}{2}(DE)\left(\boxed{}\right)\sin\boxed{}$

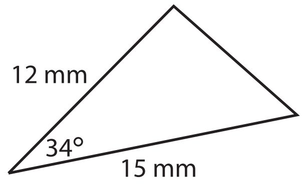
Substitute in the area formula. $\quad$ Area $= \frac{1}{2}\left(\boxed{}\right)\left(\boxed{}\right)\sin\boxed{}°$

Evaluate, rounding to the nearest tenth. $\quad$ Area $\approx \boxed{}$ in.^{2}

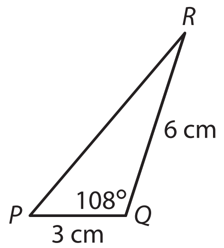
Your Turn

Find the area of each triangle to the nearest tenth.

3.



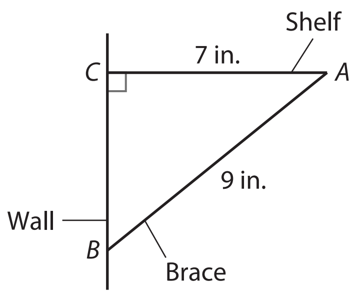
4. In $\triangle PQR$, $PQ = 3$ cm, $QR = 6$ cm, and $m\angle Q = 108°$.



Explain 2 Solving a Right Triangle

Solving a right triangle means finding the lengths of all its sides and the measures of all its angles. To solve a right triangle you need to know two side lengths or one side length and an acute angle measure. Based on the given information, choose among trigonometric ratios, inverse trigonometric ratios, and the Pythagorean Theorem to help you solve the right triangle.

A shelf extends perpendicularly 7 in. from a wall. You want to place a 9-in. brace under the shelf, as shown. To the nearest tenth of an inch, how far below the shelf will the brace be attached to the wall? To the nearest degree, what angle will the brace make with the shelf and with the wall?



(A) Find BC.

Use the Pythagorean Theorem to find the length of the third side.	$AC^2 + BC^2 = AB^2$
Substitute 7 for AC and 9 for AB.	$7^2 + BC^2 = 9^2$
Find the squares.	$49 + BC^2 = 81$
Subtract 49 from both sides.	$BC^2 = 32$
Find the square root and root.	$BC \approx 5.7$

(B) Find m$\angle A$ and m$\angle B$.

Use an inverse trigonometric ratio to find m$\angle A$. You know the lengths of the adjacent side and the hypotenuse, so use the cosine ratio.

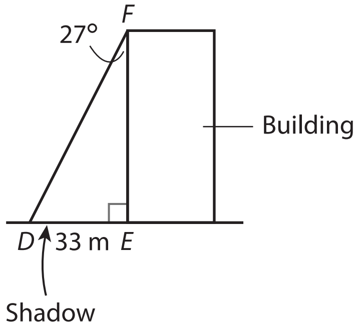
Write a cosine ratio for $\angle A$.	$\cos A = \square$
Write an inverse cosine ratio.	$m\angle A = \cos^{-1}\left(\square\right)$
Evaluate the inverse cosine ratio and round.	$m\angle A \approx \square^\circ$
$\angle\square$ and $\angle B$ are complementary.	$m\angle\square + m\angle B = 90^\circ$
Substitute $\square^\circ$ for m$\angle\square$.	$\square^\circ + m\angle B \approx 90^\circ$
Subtract $\square^\circ$ from both sides.	$m\angle B \approx \square^\circ$

Reflect

5. Is it possible to find m$\angle B$ before you find m$\angle A$? Explain.

Your Turn

A building casts a 33-m shadow when the Sun is at an angle of 27° to the vertical. How tall is the building, to the nearest meter? How far is it from the top of the building to the tip of the shadow? What angle does a ray from the Sun along the edge of the shadow make with the ground?



6. Use a trigonometric ratio to find the distance *EF*.

7. Use another trigonometric ratio to find the distance *DF*.

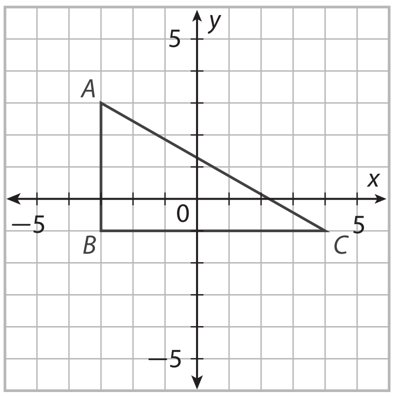
8. Use the fact that acute angles of a right triangle are complementary to find m∠*D*.

Explain 3 Solving a Right Triangle in the Coordinate Plane

You can use the distance formula as well as trigonometric tools to solve right triangles in the coordinate plane.

Example 3 **Solve each triangle.**

Ⓐ Triangle *ABC* has vertices $A(-3, 3)$, $B(-3, -1)$, and $C(4, -1)$. Find the side lengths to the nearest hundredth and the angle measures to the nearest degree.



Plot points *A*, *B*, and *C*, and draw $\triangle ABC$.

Find the side lengths: $AB = 4$, $BC = 7$

Use the distance formula to find the length of $\overline{AC}$.

$AC = \sqrt{\left(4 - (-3)\right)^2 + (-1 - 3)^2} = \sqrt{65} \approx 8.06$

Find the angle measures: $\overline{AB} \perp \overline{BC}$, so $m\angle B = 90°$.

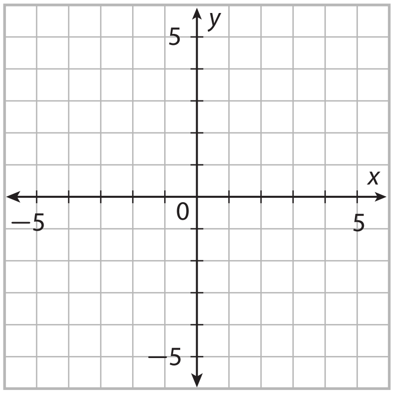
Use an inverse tangent ratio to find

$m\angle C = \tan^{-1}\left(\frac{AB}{BC}\right) = \tan^{-1}\left(\frac{4}{7}\right) \approx 30°$.

$\angle A$ and $\angle C$ are complementary, so $m\angle A \approx 90° - 30° = 60°$.

B Triangle DEF has vertices $D(-4, 3)$, $E(3, 4)$, and $F(0, 0)$. Find the side lengths to the nearest hundredth and the angle measures to the nearest degree.

Plot points D, E, and F, and draw $\triangle DEF$.



$\angle F$ appears to be a right angle. To check, find the slope

of $\overline{DF}$: $\dfrac{\square - 3}{0 - \square} = \dfrac{\square}{\square} = \square$;

slope of $\overline{\square}$: $\dfrac{\square - \square}{\square - 3} = \dfrac{\square}{\square} = \dfrac{\square}{\square}$;

so $m\angle F = \square°$.

Find the side lengths using the distance formula:

$DE = \sqrt{\left(3 - \square\right)^2 + \left(\square - 3\right)^2} = \sqrt{\square} = \square\sqrt{\square} \approx \square$,

$DF = \sqrt{\left(\square - \square\right)^2 + \left(\square - 3\right)^2} = \sqrt{\square} = \square$,

$\square = \sqrt{\left(\square - \square\right)^2 + \left(\square - 4\right)^2} = \sqrt{\square} = \square$

Use an inverse sine ratio to find $m\angle D$.

$m\angle D = \sin^{-1}\left(\dfrac{EF}{\square}\right) = \sin^{-1}\left(\dfrac{\square}{\square}\right) = \square°$

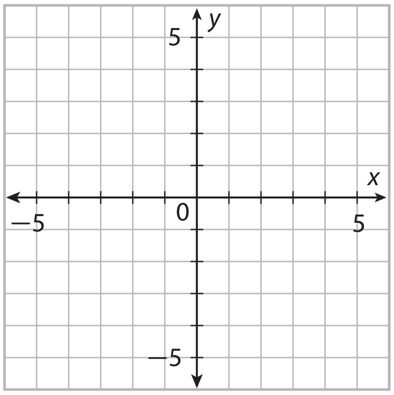
$\angle D$ and $\angle\square$ are complementary, so $m\angle\square = 90° - \square° = \square°$.

Reflect

9. How does the given information determine which inverse trigonometric ratio you should use to determine an acute angle measure?

Your Turn

10. Triangle JKL has vertices $J(3, 5)$, $K(-3, 2)$, and $L(5, 1)$. Find the side lengths to the nearest hundredth and the angle measures to the nearest degree.



Elaborate

11. Would you use the area formula you determined in this lesson for a right triangle? Explain.

12. **Discussion** How does the process of solving a right triangle change when its vertices are located in the coordinate plane?

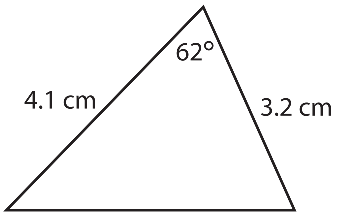
13. **Essential Question Check-In** How do you find the unknown angle measures in a right triangle?

Evaluate: Homework and Practice

• Online Homework
• Hints and Help
• Extra Practice

Find the area of each triangle to the nearest tenth.

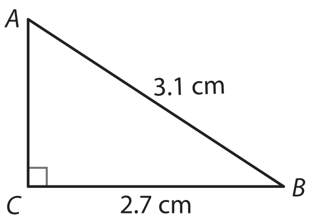
1.



2. In $\triangle PQR$, $PR = 23$ mm, $QR = 39$ mm, and $m\angle R = 163°$.

Solve each right triangle. Round lengths to the nearest tenth and angles to the nearest degree.

3.



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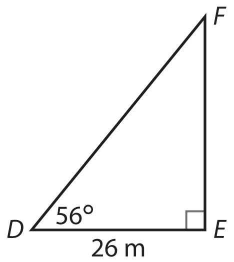
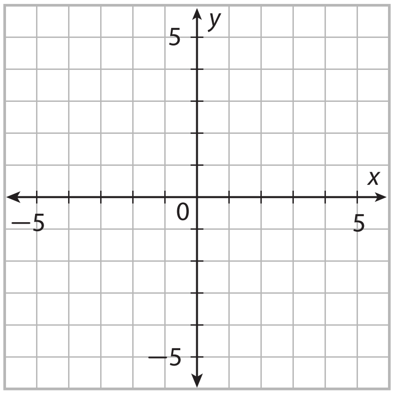


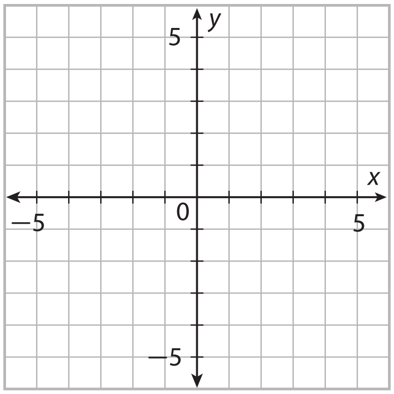
5. Right $\triangle PQR$ with $\overline{PQ} \perp \overline{PR}$, $QR = 47$ mm, and $m\angle Q = 52°$

Solve each triangle. Find the side lengths to the nearest hundredth and the angle measures to the nearest degree.

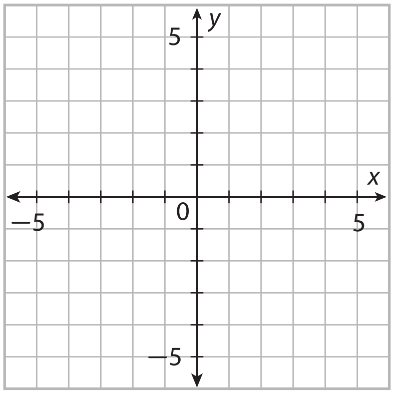
6. Triangle ABC with vertices $A(-4, 4)$, $B(3, 4)$, and $C(3, -2)$



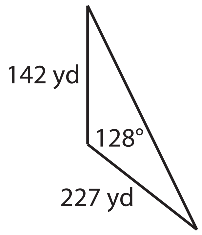
7. Triangle JKL with vertices $J(-3, 1)$, $K(-1, 4)$, and $L(6, -5)$



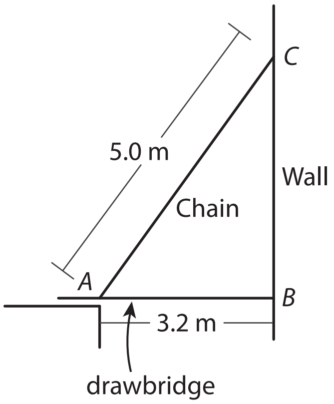
8. Triangle PQR with vertices $P(5, 5)$, $Q(-5, 3)$, and $R(-4, -2)$



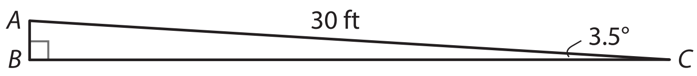
9. **Surveying** A plot of land is in the shape of a triangle, as shown. Find the area of the plot, to the nearest hundred square yards.



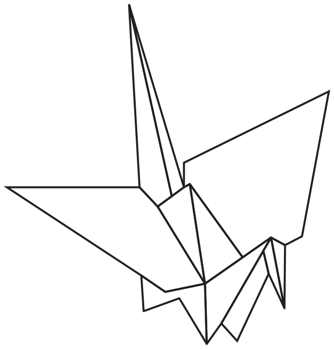
10. **History** A drawbridge at the entrance to an ancient castle is raised and lowered by a pair of chains. The figure represents the drawbridge when flat. Find the height of the suspension point of the chain, to the nearest tenth of a meter, and the measures of the acute angles the chain makes with the wall and the drawbridge, to the nearest degree.

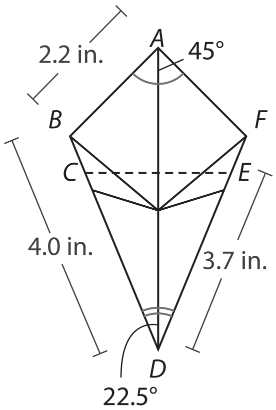


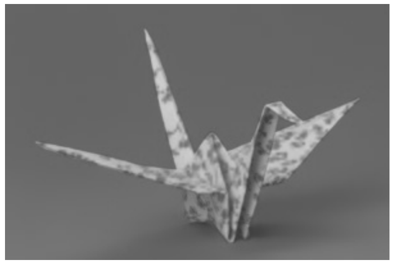
11. **Building** For safety, the angle a wheelchair ramp makes with the horizontal should be no more than 3.5°. What is the maximum height of a ramp of length 30 ft? What distance along the ground would this ramp cover? Round to the nearest tenth of a foot.



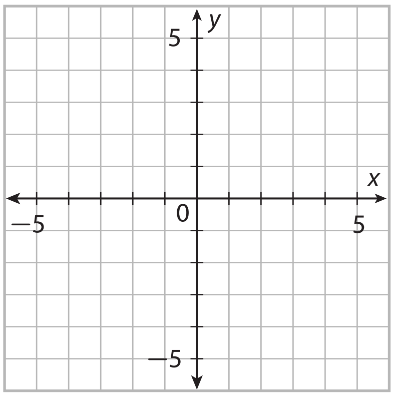
12. **Multi-Step** The figure shows an origami crane as well as a stage of its construction. The area of each wing is shown by the shaded part of the figure, which is symmetric about its vertical center line. Use the information in the figure to find the total wing area of the crane, to the nearest tenth of a square inch.





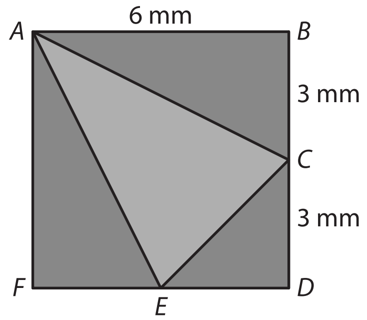


13. Right triangle $\triangle XYZ$ has vertices $X(1, 4)$ and $Y(2, -3)$. The vertex Z has positive integer coordinates, and $XZ = 5$. Find the coordinates of Z and solve $\triangle XYZ$; give exact answers.

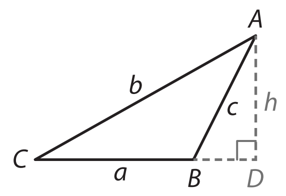


14. **Critique Reasoning** Shania and Pedro are discussing whether it is always possible to solve a right triangle, given enough information, without using the Pythagorean Theorem. Pedro says that it is always possible, but Shania thinks that when two side lengths and no angle measures are given, the Pythagorean Theorem is needed. Who is correct, and why?

15. **Design** The logo shown is symmetrical about one of its diagonals. Find the angle measures in $\triangle CAE$, to the nearest degree. (*Hint:* First find an angle in $\triangle ABC$, $\triangle CDE$ or $\triangle AEF$) Then, find the area of $\triangle CAE$, without first finding the areas of the other triangles.

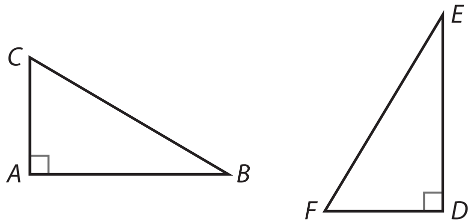


16. Use the area formula for obtuse $\angle B$ in the diagram to show that if an acute angle and an obtuse angle are supplementary, then their sines are equal.



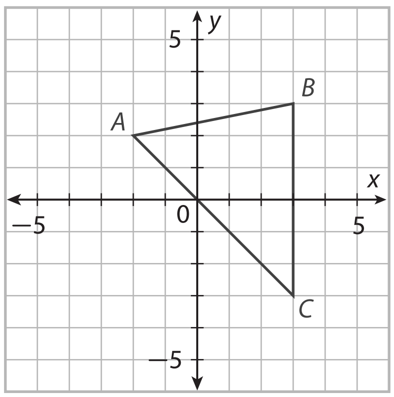
H.O.T. Focus on Higher Order Thinking

17. Communicate Mathematical Ideas The HL Congruence Theorem states that for right triangles ABC and DEF such that $\angle A$ and $\angle D$ are right angles, $\overline{BC} \cong \overline{EF}$, and $\overline{AB} \cong \overline{DE}$, $\triangle ABC \cong \triangle DEF$.

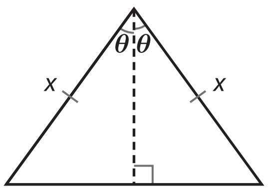


Explain, without formal proof, how solving a right triangle with given leg lengths, or with a given side length and acute angle measure, shows that right triangles with both legs congruent, or with corresponding sides and angles congruent, must be congruent.

18. Persevere in Problem Solving Find the perimeter and area of $\triangle ABC$, as exact numbers. Then, find the measures of all the angles to the nearest degree.

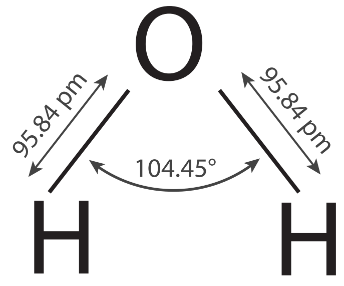


19. Analyze Relationships Find the area of the triangle using two different formulas, and deduce an expression for $\sin 2\theta$.



Lesson Performance Task

Every molecule of water contains two atoms of hydrogen and one atom of oxygen. The drawing shows how the atoms are arranged in a molecule of water, along with the incredibly precise dimensions of the molecule that physicists have been able to determine. (1 pm = 1 picometer = 10^{-12}m)



1. Draw and label a triangle with the dimensions shown.

2. Find the area of the triangle in square centimeters. Show your work.

3. Find the distance between the hydrogen atoms in centimeters. Explain your method.