15.2 Constructing Geometric Sequences

Essential Question: How do you write a geometric sequence?

⊘ Explore   Understanding Recursive and Explicit Rules for Sequences

You learned previously that an explicit rule for a sequence defines the nth term as a function of n. A recursive rule defines the nth term of a sequence in terms of one or more previous terms.

You can use what you know to identify recursive and explicit rules for sequences, and identify whether the sequences are arithmetic, geometric, or neither.

A rule for the sequence 6, 9, 13.5,... is $f(n) = 6\left(\dfrac{3}{2}\right)^{n-1}$.

(A)   The given rule is a(n) _____ rule because you do not need to know the

value of _____.

(B)   The only unknown in the expression is _____, which

represents _____.

(C)   The sequence is a(n) _____ sequence because each term is

the _____ of the previous term and $\dfrac{3}{2}$.

Reflect

1.   Discussion How can you differentiate between a geometric sequence and an arithmetic sequence?

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2.   How can you tell by looking at a function rule for a sequence whether it is a recursive rule?

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🎤 Explain 1 Constructing Recursive and Explicit Rules for Given Geometric Sequences

To write a recursive rule for a sequence, you need to know the first term, and a rule for successive terms.

Example 1 **Write a recursive rule and an explicit rule for each geometric sequence.**

(A) Makers of Japanese swords in the 1400s repeatedly folded and hammered the metal to form layers. The folding process increased the strength of the sword.

The table shows how the number of layers depends on the number of folds.

Number of Folds	n	1	2	3	4	5
Number of Layers	$f(n)$	2	4	8	16	32

To write a recursive rule, find the common ratio by calculating the ratio of consecutive terms.

$\frac{4}{2} = 2$

The common ratio r is 2.

The first term is 2, so $f(1) = 2$.

All terms after the first term are the product of the previous term and the common ratio:

$f(2) = f(1) \cdot 2, f(3) = f(2) \cdot 2, f(4) = f(3) \cdot 2, \ldots$

State the recursive rule by providing the first term and the rule for successive terms.

$f(1) = 2$

$f(n) = f(n-1) \cdot 2$ for $n \geq 2$

Write an explicit rule for the sequence by writing each term as the product of the first term and a power of the common ratio.

n	$f(n)$
1	$2(2)^0 = 2$
2	$2(2)^1 = 4$
3	$2(2)^2 = 8$
4	$2(2)^3 = 16$
5	$2(2)^4 = 32$

Generalize the results from the table: $f(n) = 2 \cdot 2^{n-1}$.

B)

n	1	2	3	4	5
f(n)	5	15	45	135	405

To write a recursive rule, find the common ratio by calculating the ratio of consecutive terms.

$$\frac{\boxed{}}{5} = \boxed{}$$

The common ratio r is _____.

The first term is _____. So, the recursive rule is:

$f(n) = f(n-1)\boxed{}$ for $n \geq 2$

Write an explicit rule for the sequence by writing each term as the product of the first term and a power of the common ratio.

n	f(n)
1	$5(3) = \boxed{}$
2	$5(3) = \boxed{}$
3	$5(3) = \boxed{}$
4	$5(3) = \boxed{}$
5	$5(3) = \boxed{}$

Generalize the results from the table: $f(n) = \boxed{}^{n-1}$.

Reflect

3. Explain why the sequence 5, 10, 20, 40, 80, … appears to be a geometric sequence.

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4. **Draw Conclusions** How can you use properties of exponents to simplify the explicit rule $f(n) = 2 \cdot 2^{n-1}$?

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Write a recursive rule and an explicit rule for each geometric sequence.

5.

n	1	2	3	4	5
$f(n)$	7	14	28	56	112

6. Write a recursive rule and an explicit rule for the geometric sequence 128, 32, 8, 2, 0.5, ...

⚙ Explain 2 Deriving the General Forms of Geometric Sequence Rules

Example 2 Use each geometric sequence to help write a recursive rule and an explicit rule for any geometric sequence. **For the general rules, the values of** n **are consecutive integers starting with 1.**

(A) 6, 24, 96, 384, 1536, …

Find the common ratio.

Numbers   **Algebra**

6, 24, 96, 384, 1536,… $f(1), f(2), f(3), f(4), f(5),...$

Common ratio $= 4$ Common ratio $= r$

Write a recursive rule.

Numbers   **Algebra**

$f(1) = 6$ and   Given $f(1)$,

$f(n) = f(n-1) \cdot 4$ for $n \geq 2$ $f(n) = f(n-1) \cdot r$ for $n \geq 2$

Write an explicit rule.

Numbers   **Algebra**

$f(n) = 6 \cdot 4^{n-1}$ $f(n) = f(1) \cdot r^{n-1}$

(B) 4, 12, 36, 108, 324,...

Find the common ratio.

Numbers

4, 12, 36, 108, 324,...

Common ratio = ☐

Write a recursive rule.

Numbers

$f(1) =$ ☐ and

$f(n) = f(n-1) \cdot$ ☐ for $n \geq$ ☐

Write an explicit rule.

Numbers

$f(n) =$ ☐ $\cdot$ ☐ $^{n-1}$

Algebra

$f(1), f(2), f(3), f(4), f(5),...$

Common ratio = ☐

Algebra

Given $f(1)$,

$f(n) = f(n-1) \cdot$ ☐ for $n \geq$ ☐

Algebra

$f(n) = f\left(☐\right) \cdot$ ☐ $^{n-1}$

> **Reflect**

7. **Discussion** The first term of a geometric sequence is 81 and the common ratio is $\frac{1}{3}$. Explain how the 4th term of the sequence can be determined.

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8. What is the recursive rule for the sequence $f(n) = 5(4)^{n-1}$?

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> **Your Turn**

Use each geometric sequence to help write a recursive rule and an explicit rule for any geometric sequence.

9. 6, 12, 24, 48, 96,...

The explicit and recursive rules for a geometric sequence can also be written in subscript notation. In subscript notation, the subscript indicates the position of the term in the sequence. a_1, a_2, and a_3 are the first, second, and third terms of a sequence respectively. In general, a_n is the nth term of a sequence.

Example 3 Write an explicit rule for each sequence using subscript notation.

(A) **Photography** The shutter speed settings on a camera form a geometric sequence where a_n is the shutter speed in seconds and n is the setting number. The fifth setting on the camera is $\frac{1}{60}$ second and the seventh setting on the camera is $\frac{1}{15}$ second.

Identify the given terms in the sequence.

The fifth setting is $\frac{1}{60}$ second, so the 5th term of the sequence is $\frac{1}{60}$.

$$a_5 = \frac{1}{60}$$

The seventh setting is $\frac{1}{15}$ second, so the 7th term of the sequence is $\frac{1}{15}$.

$$a_7 = \frac{1}{15}$$

Find the common ratio.

$a_7 = a_6 \cdot r$	Write the recursive rule for a_7.
$a_6 = a_5 \cdot r$	Write the recursive rule for a_6.
$a_7 = a_5 \cdot r \cdot r$	Substitute the expression for a_6 into the rule for a_7.
$\frac{1}{15} = \frac{1}{60} \cdot r^2$	Substitute $\frac{1}{15}$ for a_7 and $\frac{1}{60}$ for a_5.
$4 = r^2$	Multiply both sides by 60.
$2 = r$	Definition of positive square root

Find the first term of the sequence.

$a_n = a_1 \cdot r^{n-1}$	Write the explicit rule.
$\frac{1}{60} = a_1 \cdot 2^{5-1}$	Substitute $\frac{1}{60}$ for a_n, 2 for r, and 5 for n.
$\frac{1}{60} = a_1 \cdot 16$	Simplify.
$\frac{1}{960} = a_1$	Divide both sides by 16.

Write the explicit rule.

$a_n = a_1 \cdot r^{n-1}$	Write the general rule.
$a_n = \frac{1}{960} \cdot (2)^{n-1}$	Substitute $\frac{1}{960}$ for a_1 and 2 for r.

B **Viral Video** You tell a number of friends about an interesting video you saw online. Each of those friends tells the same number of friends about it. This pattern continues, and there are no repeats in the people told. The numbers of people who hear about this video through you form a geometric sequence. There are 256 people at the fourth round and 4096 people at the sixth round.

Identify the given terms in the sequence.

The 4th term of the sequence is $\boxed{}$.

$a_4 = \boxed{}$

The 6th term of the sequence is $\boxed{}$.

$a_6 = \boxed{}$

Find the common ratio.

$a_6 = \boxed{} \cdot r$ Write the recursive rule for a_6.

$a_6 = \boxed{} \cdot r$ Write the recursive rule for a_5.

$a_6 = \boxed{} \cdot \boxed{} \cdot r$ Substitute the expression for a_5 into the rule for a_6.

$\boxed{} = \boxed{} \cdot r^2$ Substitute $\boxed{}$ for a_6 and $\boxed{}$ for a_4.

$\boxed{} = r^2$ Divide both sides by 256.

$\boxed{} = r$ Definition of positive square root.

Find the first term of the sequence.

$a_n = a_1 \cdot r^{n-1}$ Write the explicit rule.

$\boxed{} = a_1 \cdot \boxed{}^{-1}$ Substitute $\boxed{}$ for a_n, $\boxed{}$ for r, and $\boxed{}$ for n.

$\boxed{} = a_1 \cdot \boxed{}$ Simplify.

$\boxed{} = a_1$ Divide both sides by $\boxed{}$.

Write the explicit rule.

$a_n = a_1 \cdot r^{n-1}$ Write the general rule.

$a_n = \boxed{} \cdot \left(\boxed{} \right)^{n-1}$ Substitute $\boxed{}$ for a_1 and $\boxed{}$ for r.

Reflect

10. Finding the common ratio in the shutter speed example involved finding a square root. Why was the negative square root not considered?

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_____

Write an explicit rule for the sequence using subscript notation.

11. The third term of a geometric sequence is $\frac{1}{27}$ and the fifth term is $\frac{1}{243}$. All the terms of the sequence are positive.

💬 Elaborate

12. **What If** Suppose you are given the terms a_3 and a_6 of a geometric sequence. How can you find the common ratio r?

_____

13. If you know the second term and the common ratio of a geometric sequence, can you write an explicit rule for the sequence? If so, explain how.

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_____

14. **Essential Question Check-In** How can you write the explicit rule for a geometric sequence if you know the recursive rule for the sequence?

_____

_____

For each geometric sequence, write a recursive rule by finding the common ratio by calculating the ratio of consecutive terms. Write an explicit rule for the sequence by writing each term as the product of the first term and a power of the common ratio.

1.

n	1	2	3	4	5
a_n	2	6	18	54	162

2.

n	1	2	3	4	5
a_n	10	3	0.9	0.27	0.081

3.

n	1	2	3	4	5
a_n	5	20	80	320	1280

4.

n	1	2	3	4	5
a_n	6	−3	1.5	−0.75	0.375

5.

n	1	2	3	4	5
a_n	9	6	4	$2\frac{2}{3}$	$1\frac{7}{9}$

6.

n	1	2	3	4	5
a_n	−12	6	−3	1.5	−0.75

7.

n	1	2	3	4	5
a_n	4	24	144	864	5184

8.

n	1	2	3	4	5
a_n	10	5	2.5	1.25	0.625

9.

n	1	2	3	4	5
a_n	3	21	147	1029	7203

10.

n	1	2	3	4	5
a_n	8	72	648	5832	52,488

11.

n	1	2	3	4	5
a_n	6	30	150	750	3750

Use the geometric sequence to help write a recursive rule and an explicit rule for any geometric sequence. For the general rules, the values of n are consecutive integers starting with 1.

**12.** 5, 15, 45, 135, 405,…

13. 10, 40, 160, 640, 2,560,…

14. 5, 10, 20, 40, 80,…

15. 18, 90, 450, 2250, 11,250,…

Write an explicit rule for each geometric sequence using subscript notation. Use a calculator and round your answer to the nearest tenth if necessary.

16. The fifth term of the sequence is 5. The sixth term is 2.5.

17. The third term of the sequence is 120. The fifth term is 76.8.

18. The fourth term of the sequence is 216. The sixth term is 96.

19. **Sports** The numbers of teams remaining in each round of a single-elimination tennis tournament represent a geometric sequence where a_n is the number of teams competing and n is the round. There are 32 teams remaining in round 4 and 8 teams in round 6.

20. **Video Games** The numbers of points that a player must accumulate to reach each next level of a video game form a geometric sequence where a_n is the number of points needed to complete level n. You need 20,000 points to complete level 3 and 8,000,000 points to complete level 5.

21. **Conservation** A state began an effort to increase the deer population. In year 2 of the effort, the deer population in a state forest was 1200. In year 4, the population was 1728.
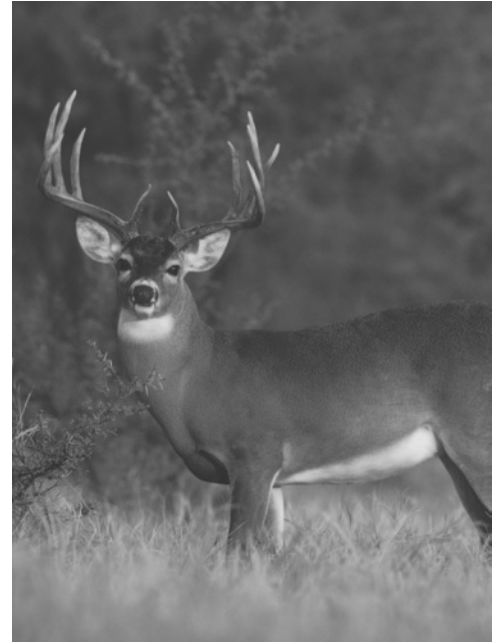
22. **Biology** The growth of a local raccoon population approximates a geometric sequence where a_n is the number of raccoons in a given year and n is the year. After 6 years there are 45 raccoons and after 8 years there are 71 raccoons.

23. **Chemistry** A chemist measures the temperature in degrees Fahrenheit of a chemical compound every hour. The temperatures approximate a geometric sequence where a_n is the temperature at a given hour, and n is the hour. At hour 4, the temperature is 70 °F and at hour 6 the temperature is 80 °F.

24. Yusuf was asked to write a recursive rule for a sequence. Which of the following is an appropriate answer? Select all that apply.

    a. $f(n) = 11(5)^{n-1}$ **b.** $f(n) = 11f(n-1), f(1) = 555$ **c.** $f(n) = f(n-1) + 15, f(1) = 36$

    d. $f(n) = 12 + 19 \cdot f(n-1)$ **e.** $f(n) = -4\left(\dfrac{2}{3}\right)^{n-1}$

25. **Multi-Step** An economist predicts that the cost of food will increase by 4% per year for the next several years.

   a. Write an explicit rule for the sequence that gives the cost $f(n)$ in dollars of a box of cereal in year n that costs \$3.20 in year 1. Justify your answer.

   b. What is the fourth term of the sequence? What does it represent in this situation? Justify your answer.
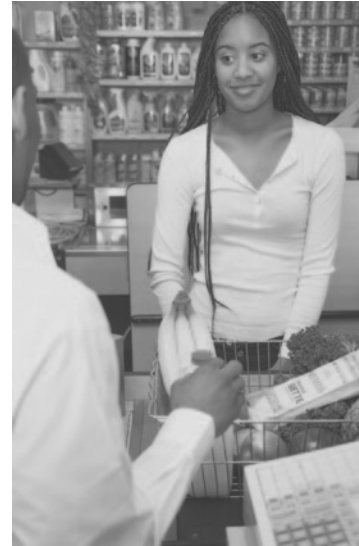
26. **Analyze Relationships** Suppose you know the 8th term of a geometric sequence and the common ratio r. How can you find the 3rd term of the sequence without writing a rule for the sequence? Explain.

27. **Explain the Error** Given that the second term of a sequence is 64 and the fourth term is 16, Francis wrote the explicit rule $a_n = 128 \cdot \left(\frac{1}{4}\right)^{n-1}$ for the sequence. Explain his error.

28. **Communicate Mathematical Ideas** Suppose you are given two terms of a geometric sequence like the ones in Example 3, except that both terms are negative. Explain how writing the explicit rule for the sequence would differ from the examples in this lesson.

Lesson Performance Task

The table shows how a population of rabbits has changed over time. Write an explicit rule for the geometric sequence described in the table. In what year will there be more than 5000 rabbits?

Time (years), n	Population, a_n
1	800
2	1200
3	1800
4	2700