22.2 Solving Equations by Completing the Square
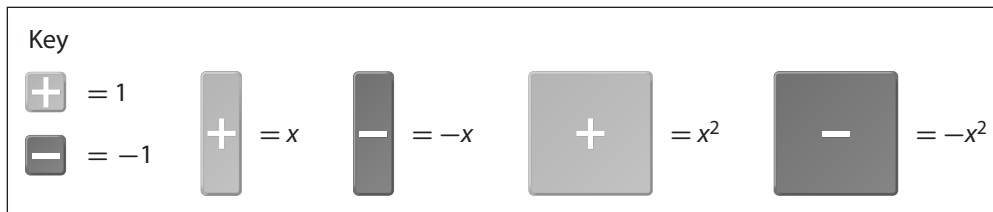
Essential Question: How can you use completing the square to solve a quadratic equation?

🧭 Explore Modeling Completing the Square

You can use algebra tiles to model a perfect square trinomial.

Key

$\boxed{+} = 1$

$\boxed{-} = -1$

$\boxed{+} = x$

$\boxed{-} = -x$

$\boxed{+} = x^2$

$\boxed{-} = -x^2$

(A) The algebra tiles shown represent the expression $x^2 + 6x$. The expression does not have a constant term, which would be represented with unit tiles. Create a square diagram of algebra tiles by adding the correct number of unit tiles to form a square.
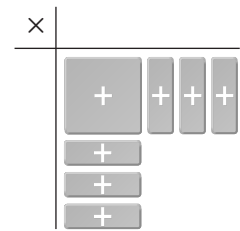
(B) How many unit tiles were added to the expression? _____

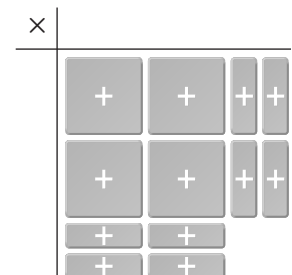**(C)** Write the trinomial represented by the algebra tiles for the complete square.

$\boxed{}x^2 + \boxed{}x + \boxed{}$

(D) It should be easily recognized that the trinomial $\boxed{}x^2 + \boxed{}x + \boxed{}$ is an example of the special case $(a + b)^2 = a^2 + 2ab + b^2$. Recall that trinomials of this form are called perfect-square trinomials. Since the trinomial is a perfect square, it can be factored into two identical binomials.

$\boxed{}x^2 + \boxed{}x + \boxed{} = \left(\boxed{}x + \boxed{}\right)^2$

(E) Refer to the algebra tiles in the diagram. What expression is represented by the tiles?

$\boxed{}x^2 + \boxed{}x$

(F) Complete the square in Step E by filling the bottom right corner with unit tiles. How many unit tiles were added to the diagram? _____

(G) Write the trinomial represented by the algebra tiles for the complete square.

$\boxed{} x^2 + \boxed{} x + \boxed{}$

(H) The trinomial is a square of a binomial. Use the algebra tiles to write the trinomial in factored form.

$\boxed{} x^2 + \boxed{} x + \boxed{} = \left(\boxed{} x + \boxed{} \right)^2$

Reflect

1. **Discussion** When using algebra tiles to model the expression $x^2 + 6x + c$, the x-tiles are divided equally, with 3 tiles on the right and bottom sides of the x^2-tile. How does the number 3 relate to the total number of x-tiles? How does the number 3 relate to the total number of unit tiles that were added?

_____

_____

2. In order to form a perfect square trinomial with the expression $x^2 + 8x + c$, how would the algebra tiles be arranged? How many unit tiles must be added? How is the number of unit tiles added related to the total number of x-tiles?

_____

_____

_____

🔧 Explain 1   Completing the Square When $a = 1$

Completing the square is a process of rewriting a quadratic expression as a perfect square trinomial so that it can be solved by taking square roots. In the Explore, the method for completing the square when $a = 1$ was modeled with algebra tiles. First, place half of the x-tiles along the right side of the x^2-tile and half underneath the tile. Then add unit tiles to fill in the rectangle started with the x^2 - and x-tiles. The number of unit tiles equals the square of the number of x-tiles on either side of the x^2-tile.

In other words, to complete the square for the expression $x^2 + bx + c$, add $\left(\frac{b}{2}\right)^2$. The perfect-square trinomial will then be $x^2 + bx + \left(\frac{b}{2}\right)^2$ and its factored form will be $\left(x + \frac{b}{2}\right)^2$.

Example 1   Complete the square to form a perfect-square trinomial.

(A) $x^2 + 4x$

$x^2 + 4x$

$b = 4$ Identify b.

$\left(\frac{4}{2}\right)^2 = 2^2 = 4$ Find $\left(\frac{b}{2}\right)^2$.

$x^2 + 4x + 4$ Add $\left(\frac{b}{2}\right)^2$ to the expression.

(B) $x^2 - 8x$

$x^2 - 8x$

$b = \boxed{}$ Identify b.

$\left(\dfrac{\boxed{}}{2}\right)^2 = \left(\boxed{}\right)^2 = \boxed{}$ Find $\left(\dfrac{b}{2}\right)^2$.

$x^2 - 8x + \boxed{}$ Add $\left(\dfrac{b}{2}\right)^2$ to the expression.

Reflect

3. When b is negative, why is the result added to the expression still positive?

_____

_____

Your Turn

4. Complete the square: $x^2 + 12x$

🔧 Explain 2   Solving $x^2 + bx + c = 0$ by Completing the Square

Completing the square can also be used to solve equations in the forms $x^2 + bx + c = 0$ or $x^2 + bx = c$.

Example 2   **Solve each equation by completing the square. Check the answers.**

(A) $x^2 - 4x = 3$

$$x^2 - 4x = 3$$

$x^2 - 4x + 4 = 3 + 4$ Add $\left(\dfrac{b}{2}\right)^2 = \left(\dfrac{-4}{2}\right)^2 = 4$ to both sides.

$(x - 2)^2 = 7$ Factor and simplify.

$x - 2 = \pm\sqrt{7}$ Take the square root of both sides.

$x - 2 = \sqrt{7}$ or $x - 2 = -\sqrt{7}$ Write and solve two equations.

$x = 2 + \sqrt{7}$ $x = 2 - \sqrt{7}$ Add 2 to both sides.

Check the answers.

$(2 + \sqrt{7})^2 - 4(2 + \sqrt{7})$ $(2 - \sqrt{7})^2 - 4(2 - \sqrt{7})$

$= 4 + 4\sqrt{7} + 7 - 8 - 4\sqrt{7}$ $= 4 - 4\sqrt{7} + 7 - 8 + 4\sqrt{7}$

$= 4 + 7 - 8 + 4\sqrt{7} - 4\sqrt{7}$ $= 4 + 7 - 8 - 4\sqrt{7} + 4\sqrt{7}$

$= 3$ $= 3$

$2 + \sqrt{7}$ and $2 - \sqrt{7}$ are both solutions of the equation $x^2 - 4x = 3$.

Ⓑ $x^2 + 16x = 36$

$$x^2 + 16x = 36$$

$$\boxed{}x^2 + \boxed{}x + \boxed{} = \boxed{} + \boxed{}$$ Add $\left(\dfrac{b}{2}\right)^2 = \boxed{}$ to both sides.

$$\left(\boxed{}x + \boxed{}\right)^2 = \boxed{}$$ Factor and simplify.

$$\boxed{}x + \boxed{} = \pm\boxed{}$$ Take the square root of both sides.

$$\boxed{}x + \boxed{} = \boxed{} \quad \text{or} \quad \boxed{}x + \boxed{} = -\boxed{}$$

$$x = \boxed{} \qquad\qquad x = \boxed{}$$

Check the answers.

$$x^2 + 16x = 36$$

$$(-18)^2 + 16 \cdot \boxed{} = 36$$

$$\boxed{} - \boxed{} = 36$$

$$\boxed{} = 36$$

$x = -18$ is/is not a solution to the equation $x^2 + 16x = 36$.

$$x^2 + 16x = 36$$

$$2^2 + 16 \cdot \boxed{} = 36$$

$$\boxed{} + \boxed{} = 36$$

$$\boxed{} = 36$$

$x = 2$ is/is not a solution to the equation $x^2 + 16x = 36$

Your Turn

Solve each equation by completing the square. Check the answers.

5. $x^2 - 10x = 11$

6. $x^2 + 6x = 2$

✏️ Explain 3 · Solving $ax^2 + bx + c = 0$ by Completing the Square When a Is a Perfect Square

When a is a perfect square, completing the square is easier than in other cases. Recall that the number of unit tiles needed is equal to the square of b divided by four times a, or $\frac{b^2}{4a}$. This is always the case when a is a perfect square.

Example 3   Solve each equation by completing the square.

Ⓐ $4x^2 - 8x = 21$

$$4x^2 - 8x = 21$$

$$\frac{(-8)^2}{4 \cdot 4} = \frac{64}{16} = 4 \qquad \text{Find } \frac{b^2}{4a}.$$

$$4x^2 - 8x + 4 = 21 + 4 \qquad \text{Add } \frac{b^2}{4a} \text{ to both sides.}$$

$$(2x - 2)^2 = 25 \qquad \text{Factor and simplify.}$$

$$2x - 2 = \pm\sqrt{25} \qquad \text{Take the square root of the both sides.}$$

$$2x - 2 = \pm 5 \qquad \text{Simplify.}$$

$$2x - 2 = 5 \quad \text{or} \quad 2x - 2 = -5 \qquad \text{Write and solve 2 equations.}$$

$$2x = 7 \quad \text{or} \quad 2x = -3 \qquad \text{Add to both sides.}$$

$$x = \frac{7}{2} \quad \text{or} \quad x = -\frac{3}{2} \qquad \text{Divide both sides by 2.}$$

Ⓑ $9x^2 + 6x = 10$

$$9x^2 + 6x = 10$$

$$\frac{\boxed{}^2}{4 \cdot \boxed{}} = \frac{\boxed{}}{\boxed{}} = \boxed{} \qquad \text{Find } \frac{b^2}{4a}.$$

$$\boxed{}x^2 + \boxed{}x + \boxed{} = 10 + \boxed{} \qquad \text{Add } \frac{b^2}{4a} \text{ to both sides.}$$

$$\left(\boxed{}x + \boxed{}\right)^2 = \boxed{} \qquad \text{Factor and simplify.}$$

$$\boxed{}x + \boxed{} = \pm\sqrt{\boxed{}} \qquad \text{Take the square root of the both sides.}$$

$$\boxed{}x + \boxed{} = \sqrt{\boxed{}} \quad \text{or} \quad \boxed{}x + \boxed{} = -\sqrt{\boxed{}} \qquad \text{Write and solve two equations.}$$

$$\boxed{}x = -\boxed{} + \sqrt{\boxed{}} \qquad \boxed{}x = -\boxed{} - \sqrt{\boxed{}} \qquad \text{Subtract } \boxed{} \text{ from both sides.}$$

$$x = \frac{-\boxed{} + \sqrt{\boxed{}}}{\boxed{}} \qquad x = \frac{-\boxed{} - \sqrt{\boxed{}}}{\boxed{}} \qquad \text{Divide both sides by } \boxed{}.$$

7. In order for the procedure used in this section to work, why does a have to be a perfect square?

_____

Your Turn

Solve each equation by completing the square.

8. $16x^2 - 16x = 5$

9. $4x^2 + 12x = 5$

⚙ Explain 4   Solving $ax^2 + bx + c = 0$ by Completing the Square When a Is Not a Perfect Square

When the leading coefficient a is not a perfect square, the equation can be transformed by multiplying both sides by a value such that a becomes a perfect square.

Example 4   **Solve each equation by completing the square.**

Ⓐ $2x^2 - 6x = 5$

Since the coefficient of x^2 is 2, which is not a perfect square, multiply both sides by a value so the coefficient will have a perfect square. In this case, use 2.

$$2x^2 - 6x = 5$$

$$2(2x^2 - 6x) = 2(5)$$ Multiply both sides by 2.

$$4x^2 - 12x = 10$$ Simplify.

$$\frac{(-12)^2}{4 \cdot 4} = \frac{144}{16} = 9$$ Find $\frac{b^2}{4a}$.

$$4x^2 - 12x + 9 = 10 + 9$$ Add $\frac{b^2}{4a}$ to both sides.

$$(2x - 3)^2 = 19$$ Factor and simplify.

$$2x - 3 = \pm\sqrt{19}$$ Take the square root of the both sides.

$$2x - 3 = \sqrt{19} \quad \text{or} \quad 2x - 3 = -\sqrt{19}$$ Write and solve 2 equations.

$$2x = 3 + \sqrt{19} \qquad 2x = 3 - \sqrt{19}$$ Add to both sides.

$$x = \frac{3 + \sqrt{19}}{2} \qquad x = \frac{3 - \sqrt{19}}{2}$$ Divide both sides by 2.

(B) $3x^2 + 3x = 16$

Since the coefficient of x^2 is _____, which is not a perfect square, multiply both sides by a

value so the coefficient will have a perfect square. In this case, use _____.

$$3x^2 + 3x = 16$$

$$\boxed{}(3x^2 + 2x) = \boxed{}(16) \qquad\qquad \text{Multiply both sides by } \boxed{}.$$

$$\boxed{}\,x^2 + \boxed{}\,x = \boxed{} \qquad\qquad \text{Simplify.}$$

$$\frac{\boxed{}^2}{4 \cdot \boxed{}} = \frac{\boxed{}}{\boxed{}} = \boxed{} \qquad\qquad \text{Find } \frac{b^2}{4a}.$$

$$\boxed{}\,x^2 + \boxed{}\,x + \boxed{} = \boxed{} + \boxed{} \qquad\qquad \text{Add } \frac{b^2}{4a} \text{ to both sides.}$$

$$\left(\boxed{}\,x + \boxed{}\right)^2 = \boxed{} \qquad\qquad \text{Factor and simplify.}$$

$$\boxed{}\,x + \boxed{} = \pm\sqrt{\boxed{}} = \pm\boxed{} \qquad\qquad \text{Take the square root of the both sides.}$$

$$\boxed{}\,x + \boxed{} = \boxed{} \quad \text{or} \quad \boxed{}\,x + \boxed{} = -\boxed{} \qquad \text{Write and solve two equations.}$$

$$\boxed{}\,x = \boxed{} \qquad\qquad\qquad \boxed{}\,x = -\boxed{} \qquad \text{Subtract } \boxed{} \text{ from both sides.}$$

$$x = \boxed{} \qquad\qquad\qquad\quad x = -\boxed{} \qquad \text{Divide both sides by } \boxed{}.$$

Reflect

10. Consider the equation $2x^2 + 11x = 12$. Why is 2 the best value by which to multiply both sides of the equation before completing the square?

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_____

Your Turn

Solve each equation by completing the square.

11. $\frac{1}{2}x^2 + 3x = 14$

12. $2x^2 - 4x = 16$

⚙ Explain 5 Modeling Completing the Square for Quadratic Equations

Completing the square can be useful when solving problems involving quadratic functions, especially if the function cannot be factored. In these cases, complete the square to rewrite the function in vertex form: $f(x) = a(x - h)^2 + k$. Completing the square in this situation is similar to solving equations by completing the square, but instead of adding a term to both sides of the equation, you will both add and subtract it from the function's rule.

Recall that the height of an object moving under the force of gravity, with no other forces acting on it, can be modeled by the quadratic function $h = -16t^2 + vt + s$, where t is the time in seconds, v is the initial vertical velocity, and s is the initial height in feet.

Example 5 **Write a function in standard form for each model. Then, rewrite the equation in vertex form and solve the problem. Graph the function on a graphing calculator and find the *x*-intercepts and maximum value of the graph.**

Ⓐ **Sports** A baseball is thrown from a height of 5 feet. If the person throws the baseball at a velocity of 30 feet per second, what will be the maximum height of the baseball? How long will it take the baseball to hit the ground?

The function for this situation is $h = -16t^2 + 30t + 5$.

Complete the square to find the vertex of the function's graph.

$$h = -16\left(t^2 - \frac{30}{16}t\right) + 5 \qquad\qquad \text{Factor out } -16.$$

$$\frac{\left(-\frac{30}{16}\right)^2}{4} = \frac{\frac{900}{256}}{4} = \frac{900}{256} \cdot \frac{1}{4} = \frac{225}{256} \qquad\qquad \text{Find } \frac{b^2}{4}.$$

$$h = -16\left(t^2 - \frac{30}{16}t + \frac{225}{256} - \frac{225}{256}\right) + 5 \qquad\qquad \text{Complete the square.}$$

$$h = -16\left(\left(t - \frac{15}{16}\right)^2 - \frac{225}{256}\right) + 5 \qquad\qquad \text{Factor the perfect-square trinomial.}$$

$$h = -16\left(t - \frac{15}{16}\right)^2 + \frac{225}{16} + 5 \qquad\qquad \text{Distribute the } -16.$$

$$h = -16\left(t - \frac{15}{16}\right)^2 + \frac{305}{16} \qquad\qquad \text{Combine the last two terms.}$$

The coordinates of the vertex are $\left(\frac{15}{4}, \frac{305}{16}\right)$, or about (3.75, 19.06). The maximum height will be about 19 feet.

The graph of the function confirms the vertex at about (3.75, 19.06). The *t*-intercept at about 2.03 indicates that the baseball will hit the ground after about 2 seconds.

Maximum
X=.93750201 Y=19.0625

Zero
X=2.0290156 Y=0
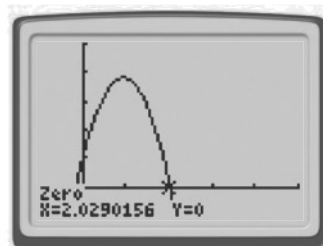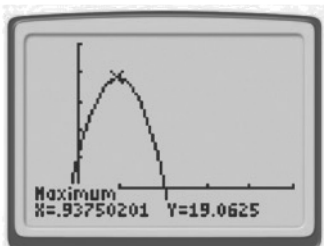
B **Sports** A person kicks a soccer ball from the ground with an initial upward velocity of 16 feet per second. What is the maximum height of the soccer ball? When will the soccer ball hit the ground?

An equation for this situation is $h = -16t^2 + \boxed{}\, t + \boxed{}$.

Complete the square to find the vertex of the function's graph.

$h = -16t^2 + \boxed{}\, t + \boxed{}$

$h = \boxed{}\left(\boxed{}\, t^2 - \boxed{}\, t\right) + \boxed{}$ Factor out $\boxed{}$.

$h = \boxed{}\left(\boxed{}\, t^2 - \boxed{}\, t + \dfrac{\boxed{}}{\boxed{}} - \dfrac{\boxed{}}{\boxed{}}\right) + 0$ Complete the square.

$h = \boxed{}\left(\left(\boxed{}\, t - \dfrac{\boxed{}}{\boxed{}}\right)^2 - \dfrac{\boxed{}}{\boxed{}}\right) + 0$ Factor the perfect-square trinomial.

$h = \boxed{}\left(\boxed{}\, t - \dfrac{\boxed{}}{\boxed{}}\right)^2 + \dfrac{\boxed{}}{\boxed{}} + 0$ Distribute the -16.

$h = \boxed{}\left(\boxed{}\, t - \dfrac{\boxed{}}{\boxed{}}\right)^2 + \boxed{}$ Combine the last two terms.

The coordinates of the vertex are $\left(\dfrac{\boxed{}}{\boxed{}},\ \boxed{}\right)$.

The soccer ball will be at its highest when it is at its vertex, or at $\boxed{}$ feet.

The graph of the function confirms the vertex at (0.5, 4). The x-intercept at 1 indicates that the ball will hit the ground after 1 second.
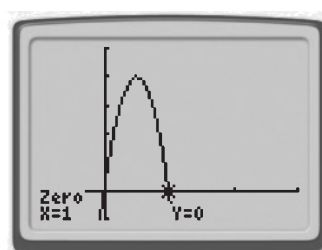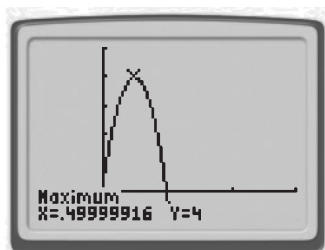
Your Turn

13. **Physics** A person standing at the edge of a cliff 48 feet tall throws a ball up and just off the cliff with an initial upward velocity of 8 feet per second. What is the maximum height of the ball? When will the ball hit the ground?

14. When $b > 0$, the perfect square-trinomial of the expression $x^2 + bx$ is $x^2 + bx + \left(\frac{b}{2}\right)^2$. What is the perfect-square trinomial when $b < 0$? Does the sign of the constant change? Why or why not?

_____

_____

15. Essential Question Check-In What is the first step in completing the square to solve a quadratic equation of the form $ax^2 + bx = c$?

_____

_____

⭐ Evaluate: Homework and Practice

Complete the square to form a perfect-square trinomial.

1. $x^2 + 26x$

2. $x^2 - 18x$

3. $x^2 - 2x$

4. $x^2 - 24x$

Solve each equation by completing the square. Check the answers.

5. $x^2 + 8x = 33$

6. $x^2 - 6x = 8$

7. $x^2 + 12x = 5$

8. $x^2 - 14x = 95$

Solve each equation by completing the square.

9. $9x^2 + 12x = 32$

10. $4x^2 + 20x = 2$

11. $16x^2 - 32x = 65$

12. $9x^2 - 24x = 1$

13. $\frac{1}{2}x^2 + 4x = 10$

14. $3x^2 - 4x = 20$

15. $2x^2 + 14x = 4$

16. $\frac{1}{2}x^2 - 5x = 18$

Projectile Motion Write an equation for each model, rewrite the equation into vertex form, and solve the problem. Then graph the function on a graphing calculator and state the *x*-intercepts of the graph.

17. **Sports** A person kicks a ball from the ground into the air with an initial upward velocity of 8 feet per second. What is the maximum height of the ball? When will the ball hit the ground?

18. **Physics** A person reaching out to the edge of a building ledge 85 feet off the ground flicks a twig up and off the ledge with an initial upward velocity of 11 feet per second. What is the maximum height of the twig? When will the twig hit the ground?

19. **Volleyball** A volleyball player hits a ball from a height of 5 feet with an initial vertical velocity of 16 feet per second. What is the maximum height of the volleyball? Assuming it is not hit by another player, when will the volleyball hit the ground?

20. **Lacrosse** A lacrosse player throws a ball into the air from a height of 8 feet with an initial vertical velocity of 32 feet per second. What is the maximum height of the ball? When will the ball hit the ground?

21. Identify the value of a in each equation of the form $ax^2 + bx + c = 0$.
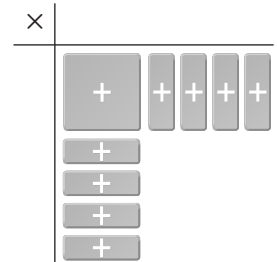
    a. $11x^2 + 2x = 4$

    b. $4x^2 + 5 = 0$

    c. $3x^3 = 7$

    d. $5x^2 + 11x = 1$

    e. $3x^2 = 5$

22. The diagram represents the expression $x^2 + 8x$. Use algebra tiles to model completing the square. Then write the perfect square trinomial expression.

23. Explain the Error A student was instructed to solve the equation $x^2 + 4x = 77$ and produced the following work. Explain the student's error. What is the correct solution?

$$x^2 + 4x = 77$$
$$x^2 + 4x + 4 = 77 + 4$$
$$(x + 2)^2 = 81$$
$$x + 2 = 9$$
$$x = 7$$

24. Justify Reasoning Will the equation $x^2 + 6x = -10$ produce an answer that is a real number after the square is completed? Explain.

25. Draw Conclusions When solving a quadratic model, why are some solutions considered extraneous? Is this always the case, or can some quadratic models have two solutions?

Lesson Performance Task

An architect is designing the lobby of a new office building. The company that hired her has reclaimed a large quantity of stone floor tiles from the building previously on the site and wishes to use them to tile the lobby. The tiled area needs to be a rectangle 18 feet longer than it is wide to incorporate an information desk. The table below shows the types of tile available, the general color of the tile, and the area that can be covered by the tile.

Stone	Color	Area in Square Feet
Marble	Cream	175
Marble	Cream with gold flecks	115
Marble	Black	648
Marble	White with black flecks	360
Slate	Gray	280
Slate	Gray with blue gray regions	243
Travertine	Caramel	208
Travertine	Latte	760
Adoquin	Dark gray with black regions	319
Adoquin	Light gray with darker gray regions	403
Limestone	Pewter	448
Limestone	Beige	544

Design the lobby using at least all of one type of tile. You can add additional types of tiles to create patterns in the floor. For this exercise, you can decide on the dimensions of the tiles in order to make any pattern you wish. What are the dimensions of the tiled area for your design?